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Finitary Topos for Locally Finite, Causal and Quantal Vacuum Einstein Gravity[1]

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The pentalogy (Mallios, A. and Raptis, I. (2001). *International Journal of Theoretical Physics* **40**, 1885; Mallios, A. and Raptis, I. (2002). *International Journal of Theoretical Physics* **41**, 1857; Mallios, A. and Raptis, I. (2003). *International Journal of Theoretical Physics* **42**, 1479; Mallios, A. and Raptis, I. (2004). 'paper-book'/research monograph); I. Raptis (2005). *International Journal of Theoretical Physics* (to appear) is brought to its categorical climax by organizing the curved finitary spacetime sheaves of quantum causal sets involved therein, on which a finitary (:locally finite), singularity-free, background manifold independent and geometrically prequantized version of the gravitational vacuum Einstein field equations were seen to hold, into a topos structure $\mathfrak{D}\mathfrak{T}_{fcq}$. We show that the category of finitary differential triads $\mathfrak{D}\mathfrak{T}_{fcq}$ is a finitary instance of an elementary topos proper in the original sense due to Lawvere and Tierney. We present in the light of Abstract Differential Geometry (ADG) a Grothendieck-type of generalization of Sorkin's finitary substitutes of continuous spacetime manifold topologies, the latter's topological refinement inverse systems of locally finite coverings and their associated coarse graining sieves, the upshot being that $\mathfrak{D}\mathfrak{T}_{fcq}$ is also a finitary example of a Grothendieck topos. In the process, we discover that the subobject classifier Ω_{fcq} of $\mathfrak{D}\mathfrak{T}_{fcq}$ is a Heyting algebra type of object, thus we infer that the internal logic of our finitary topos is intuitionistic, as expected. We also introduce the new notion of 'finitary differential geometric morphism' which, as befits ADG, gives a differential geometric slant to Sorkin's purely topological acts of refinement (:coarse graining). Based on finitary differential geometric morphisms regarded as natural transformations of the relevant sheaf categories, we observe that the functorial ADG-theoretic version of the principle of general covariance of General Relativity is preserved under topological refinement. The paper closes with a thorough discussion of four future routes we could

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take in order to further develop our topos-theoretic perspective on ADG-gravity along certain categorical trends in current quantum gravity research.

KEY WORDS: quantum gravity; causal sets; quantum logic; differential incidence algebras of locally finite partially ordered sets; abstract differential geometry; sheaf theory; category theory; topos theory.

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1. PROLOGUE CUM PHYSICAL MOTIVATION: THE PAST AND THE PRESENT

In the past $\mathfrak{a}$ $\mathfrak{A}$ decade or so, we have witnessed vigorous activity in various applications of categorical—in particular, (pre)sheaf and topos-theoretic (MacLane and Moerdijk, 1992)—ideas to Quantum Theory (QT) and Quantum Gravity (QG).

With respect to QT proper, topos theory appears to be a suitable and elegant framework in which to express the non-objective, non-classical (i.e., non-Boolean), so-called 'neo-realist' (i.e., intuitionistic), and contextual underpinnings of the logic of (non-relativistic) Quantum Mechanics (QM), as manifested for example by the Kochen-Specker theorem in standard quantum logic (Butterfield and Isham, 1998, 1999; Butterfield *et al.*, 2000; Butterfield and Isham, 2000; Rawling and Selesnick, 2000). Recently, Isham *et al.*'s topos perspective on the Kochen-Specker theorem and the Boolean algebra-localized (:contextualized) logic of QT has triggered research on applying category-theoretic ideas to the 'problem' of non-trivial localization properties of quantum observables (Zafiris, 2001). Topos theory has also been used to reveal the intuitionistic colors of the logic underlying the 'non-instrumentalist,' non-Copenhagean, 'quantum state collapse-free' consistent histories approach to QM (Isham, 1997).

At the same time, topos theory has also been applied to General Relativity (GR), especially by the Siberian school of 'toposophers' (Guts, 1991, 1995a,b; Guts and Demidov, 1993; Guts and Grinkevich, 1996; Grinkevich, 1996). Emphasis here is placed on using the intuitionistic-type of internal logic of a so-called '*formal smooth topos*,' which is assumed to replace the (category of finite-dimensional) smooth spacetime manifold(s) of GR, in order to define a new kind of differential geometry more general than the Classical (i.e., from a logical standpoint, Boolean topos **Set**-based) Differential Geometry (CDG) of finite-dimensional differential (:$\mathcal{C}^\infty$-smooth) manifolds. The tacit assumption here is that the standard kinematical structure of GR—the background pseudo-Riemannian smooth spacetime manifold—is basically (i.e., when stripped of its topological, differential and smooth Lorentzian metric structures) a classical point-set continuum living in the topos **Set** of 'constant' sets, with its 'innate' Boolean (:classical) logic (Goldblatt, 1984; MacLane and Moerdijk, 1992). This new 'intuitionistic differential calculus' pertains to the celebrated Synthetic Differential Geometry (SDG) of Kock and

Lawvere (Kock, 1981; Lavendhomme, 1996), in terms of which the differential equations of gravity (:Einstein equations) are then formulated in a 'formal smooth manifold.' A byproduct of this perspective on gravity is that the causal structure (:'causal topology') of the pointed spacetime continuum of GR is also revised, being replaced by an axiomatic scheme of 'pointless regions' and coverings for them recalling Grothendieck's pioneering work on generalized topological spaces called *sites* and their associated sheaf categories (:topoi), which culminated in the study of new, abstract (sheaf) cohomology theories in modern algebraic geometry (MacLane and Moerdijk, 1992).

Arguably however, the ultimate challenge for theoretical physics research in the new millennium is to arrive at a conceptually sound and calculationally sensible (i.e., finite) QG—the traditionally supposed (and expected!) marriage of QT with GR. Here too, category, (pre)sheaf and topos theory has been anticipated to play a central role for many different reasons, due to various different motivations, and with different aims in mind, depending on the approach to QG that one favors (Crane, 1995; Trifonov, 1995; Butterfield and Isham, 2000; Markopoulou, 2000; Isham, 2003a; Raptis, 2001c; Isham, 2003b, 2004a,b, 2005; Kato, 2004, 2005; Raptis, 2001a,c, 2003).

Akin to the present work is the recent paper of Christensen and Crane on so-called '*causal sites*' (causites) (Christensen and Crane, 2004). Like the Novosibirsk endeavors in classical GR mentioned above, this is an axiomatic looking scheme based on Grothendieck-type of 2-categories (:2-sites) in which the topological and causal structure of spacetime are intimately entwined and, when endowed with some suitable finiteness conditions, appear to be well prepared for quantization using combinatory-topological state-sum models coming from Relativistic Spin Networks and Topological Quantum Field Theory (Barrett and Crane, 1997). Ultimately, the theory aspires to lead to a finite theory of quantum spacetime geometry and QGR in a point-free 2-topos theoretic setting.

In the present paper too, we extend by topos-theoretic means previous work on applying Mallios' purely algebraic (:sheaf-theoretic) and background differential manifold independent Abstract Differential Geometry (ADG) (Mallios, 1998a,b, 2005b) towards formulating a finitary (:locally finite), causal and quantal, as well as singularities-*cum*-infinities free, version of Lorentzian (vacuum) Einstein gravity (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004). This extension is accomplished by organizing the curved finitary spacetime sheaves (finsheaves) of quantum causal sets (qausets) involved therein, on which a finitary, singularities and infinities-free, background manifold independent and geometrically (pre)quantized version of the gravitational (vacuum) Einstein field equations were seen to hold, into a '*finitary topos*' (fintopos) structure $\mathfrak{DT}_{fcq}$.[4]

[4] As in the previous pentalogy (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004), the subscript 'fcq' is an acronym standing for (f)initary, (c)ausal and (q)uantal.

The key observation supporting this topos organization of the said finsheaves is that the category $\mathfrak{D}\mathfrak{T}_{fcq}$ of finitary differential triads (fintriads)—*the* basic structural units on which our application of ADG to the finitary spacetime regime rests—is a finitary instance of an *elementary topos* (ET) in the original sense due to Lawvere and Tierney (MacLane and Moerdijk, 1992). This result is a straightforward one coming from recent thorough investigations of Papatriantafillou about the general categorical properties of the (abstract) category of differential triads $\mathfrak{D}\mathfrak{T}$ (Papatriantafillou, 2000, 2001, 2003a,b, 2004), of which $\mathfrak{D}\mathfrak{T}_{fcq}$ is a concrete and full subcategory.

There is also another way of showing that $\mathfrak{D}\mathfrak{T}_{fcq}$ is a topos. From the finitary stance that we have adopted throughout our applications of ADG-theoretic ideas to spacetime and gravity (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004), we will show that $\mathfrak{D}\mathfrak{T}_{fcq}$ is a finitary example of a *Grothendieck topos* (GT) (MacLane and Moerdijk, 1992). This arises from a general, Grothendieck-type of perspective on the finitary (open) coverings and their associated locally finite partially ordered set (poset) substitutes of continuous (spacetime) manifolds originally due to Sorkin (1991). The main structures involved here are what one might call '*covering coarse graining sieves*' adapted to the said finitary open covers (fincovers) and their associated locally finite posets. These finitary sieves (finsieves) are easily seen to define (:generate) a *Grothendieck topology* on the poset of all open subsets of the topological spacetime manifold X, which, in turn, when regarded as a poset category, is turned into a *site*—i.e., a category endowed with a *Grothendieck topology* (MacLane and Moerdijk, 1992). Then, the well known result of topos theory is evoked, namely, that the collection $\mathfrak{D}\mathfrak{T}_{fcq}$ of all the said finsheaves over this site is a finitary instance of a GT. Of course, it is a general result that every GT is an ET (MacLane and Moerdijk, 1992), thus $\mathfrak{D}\mathfrak{T}_{fcq}$ qualifies as both.[5]

Much in the same way that the locally finite posets in Sorkin (1991) were regarded as finitary substitutes of the continuous topology of the topological spacetime manifold X and, similarly, the finsheaves in Raptis (2000b) as finitary replacements of the sheaf $\mathcal{C}^0_X$ of continuous functions on X, $\mathfrak{D}\mathfrak{T}_{fcq}$ may be viewed as a finitary approximation of the elementary-Grothendieck topos (EGT) $\mathcal{S}\mathbf{hv}^0(X)$—the category of sheaves of (rings of) continuous functions over the base $\mathcal{C}^0$-manifold X (MacLane and Moerdijk, 1992). Moreover, since the construction

Occasionally we shall augment this 3-letter acronym with a fourth letter, 'v,' standing for (v)acuum. The general ADG-theoretic perspective on (vacuum Einstein) gravity may be coined '*ADG-gravity*' for short (Raptis, 2005; Mallios and Raptis, 2004). *In toto*, the theory propounded in Mallios and Raptis (2003), Raptis (2005), Mallios and Raptis (2004), and topos-theoretically extended herein, may be called '$fcqv$-ADG-gravity.'

[5] For similar Grothendieck-type of ideas in an ADG-theoretic setting, but with different physico-mathematical motivations and aims, the reader is referred to a recent paper by Zafiris (2004), which builds on the aforementioned work on algebraic quantum observables' localizations (Zafiris, 2001).

of our fintopos employs the basic ADG concepts and technology, $\mathfrak{DT}_{fcq}$ has not only topological, but also *differential geometric* attributes and significance, and thus it may be thought of as a finitary substitute of the category $\mathcal{M}an$ of finite-dimensional differential manifolds—a category that *cannot* be viewed as a topos proper. As we shall argue in the present paper, this is just one instance of the categorical versatility and import of ADG.

Furthermore, in a technical sense, since the EG fintopos $\mathfrak{DT}_{fcq}$ is manifestly (i.e., by construction) *finitely generated*, it is both *coherent* and *localic* (MacLane and Moerdijk, 1992). The underlying locale is the usual lattice of open subsets of the pointed, base topological manifold X that Sorkin initially considered in Sorkin (1991). This gives us important clues about what is the *subobject classifier* (MacLane and Moerdijk, 1992) Ω_{fcq} of $\mathfrak{DT}_{fcq}$. Also, being coherent, $\mathfrak{DT}_{fcq}$ has enough points (MacLane and Moerdijk, 1992). Indeed, these are the points (of X) that Sorkin initially 'blew up' or 'smeared out' by open subsets about them, being physically motivated by the observation that a point is an (operationally) 'ideal' entity with pathological (:'singular') behavior in GR. Parenthetically, and from a physical viewpoint, the ideal (i.e., non-pragmatic) character of spacetime points is reflected by the apparent theoretical impossibility to localize physical fields over them. Indeed, as also noted in Sorkin (1995), a conspiracy between the equivalence principle of GR and the uncertainty principle of QM appears to prohibit the infinite point-localization of the gravitational field in the sense that the more one tries to localize (:measure) the gravitational field, the more (microscopic) energy-mass-momentum probes one is forced to use, which in turn produce a gravitational field strong enough to perturb uncontrollably and without bound the original field that one initially set out to measure. In geometrical space-time imagery, one cannot localize the gravitational field more sharply than a so-called Planck length-time (in which both the quantum of action $\hbar$ and Newton's gravitational constant G are involved) without creating a black hole, which fuzzies or blurs out things so to speak. Thus, Sorkin substituted points by 'regions' (:open sets) about them, hence also, effectively, the pointed X—with the usual Euclidean $\mathcal{C}^0$-topology "*carried by its points*" (Sorkin, 1991)—was replaced by the 'pointless locale' (MacLane and Moerdijk, 1992) of its open subsets. Of course, Sorkin also provided a mechanism—technically, a projective limit procedure—for recovering (the ideal points of) the locally Euclidean continuum X from an inverse system of locally finite open covers and the finitary posets associated with them. In the end, the pointed X was recovered from the said inverse system as a dense subset of closed points of the system's projective limit space. Physically, the inverse limit procedure was interpreted as the act of topological refinement, as follows: as one employs finer and finer (:'smaller' and 'smaller') open sets to cover X (:fincover refinement), at the limit of infinite topological refinement, one effectively (i.e., modulo Hausdorff reflection (Kopperman and Wilson, 1997)) recovers the 'classical' pointed topological continuum X.

Back to our EG fintopos. In $\mathfrak{DT}_{fcq}$ we represent the aforementioned acts of topological refinement (:'topological coarse graining') of the covering finsieves and their associated finsheaves involved by '*differential geometric morphisms*'. This is a new, finitary ADG-theoretic analogue of the fundamental notion of *geometric morphism* in topos theory (MacLane and Moerdijk, 1992). This definition of differential geometric morphism essentially rests on a main result of Papatriantafillou (Papatriantafillou, 2000, 2001, 2003a,b) that a continuous map f between topological spaces (in our case, finitary poset substitutes of the topological continuum) gives rise to a pair of maps (or, categorically speaking, *adjoint functors*) (f_*, f^*) that transfer backwards and forward (between the base finitary posets) the differential structure encoded in the fintriads that the finsheaves (of incidence algebras on Sorkin's finitary posets) define. This is just one mathematical aspect of the *functoriality* of our ADG-based constructions, but physically it also supports our ADG-theoretic generalization of the Principle of General Covariance (PGC) of the manifold based GR expressed in our scheme via *natural transformations* between the relevant functor (:structure sheaf) categories within $\mathfrak{DT}$ (Mallios and Raptis, 2003, 2004). *In summa*, we will observe that general covariance, as defined abstractly in $fcqv$-ADG-gravity, is 'preserved' under the said differential geometric morphisms associated with Sorkin's acts of topological refinement.

More on the physics side, but quite heuristically, having established that $\mathfrak{DT}_{fcq}$ is a topos—a mathematical universe in which geometry and logic are closely entwined (MacLane and Moerdijk, 1992), we are poised to explore in the future deep connections between the (quantum) *logic* and the (differential) *geometry* of the vector and algebra finsheaves involved in the $fcqv$ Einstein-Lorentzian ADG-gravity. To this end, we could invoke finite dimensional, irreducible (Hilbert space) matrix representations H of the incidence algebras dwelling in the stalks of the finsheaves defining the fintriads in $\mathfrak{DT}_{fcq}$, and group them into *associated Hilbert finsheaves* $\mathcal{H}$ (Vassiliou, 1994, 1999, 2000, 2005). Accordingly, via the associated (:representation) sheaf functor (MacLane and Moerdijk, 1992; Mallios, 1998a; Vassiliou, 2005), we can organize the latter into the '*associated Hilbert fintopos*' $\mathfrak{H}_{fcq}$. The upshot of these investigations could be the identification, by using the abstract sheaf cohomological machinery of ADG and the semantics of geometric prequantization formulated *à la* ADG (Mallios, 1998b, 1999, 2004; Mallios and Raptis, 2002), of what we coin a '*quantum logical curvature*' form-like object $\mathfrak{R}$ in $\mathfrak{DT}_{fcq}$ and its representation Hilbert fintopos $\mathfrak{H}_{fcq}$. $\mathfrak{R}$ has dual action and interpretation in $(\mathfrak{DT}_{fcq}, \mathfrak{H}_{fcq})$. From a differential geometric (gravitational) standpoint (in $\mathfrak{DT}_{fcq}$), $\mathfrak{R}$ marks the well known obstruction to defining global (inertial) frames (observers) in GR. This manifests itself in the fact that the 'curved' finsheaves of $\mathfrak{DT}_{fcq}$ do not admit *global elements*—i.e., global sections. From a quantum-theoretic (logical) one (in $\mathfrak{H}_{fcq}$), $\mathfrak{R}$ represents the equally well known blockage to assigning values 'globally' to (incompatible) physical quantities in QT—the key feature of the 'warped,' 'twisted,' contextual (:Boolean subalgebras' localized), neorealist logic of quantum mechanics (Butterfield and Isham, 1998, 1999; Butterfield *et al.*, 2000; Rawling and Selesnick, 2000).

Accordingly, we envisage abstract '*sheaf cohomological quantum commutation relations*' between certain characteristic forms classifying the vector and algebra (fin)sheaves involved as the *raison d'être* of the noted obstruction(s), similarly to how in standard quantum mechanics the said inability to assign global values to physical quantities is due to the Heisenberg relations between incompatible observables such as position and momentum. In fact, as we shall see in the sequel, the 'forms' defining the characteristic classes (of vector sheaves) in ADG, and engaging into the abstract algebraic commutation relations to be proposed, have analogous (albeit, abstract) interpretation as 'position' and 'momentum' maps in the physical semantics of geometrically prequantized ADG-field theory—in particular, as the latter is applied to gravity (classical and/or quantum ADG-gravity).

The paper is organized as follows: in the next section we recall some basic properties of the abstract category $\mathfrak{DT}$ of differential triads as investigated recently by Papatriantafillou (Papatriantafillou, 2000, 2001, 2003a,b, 2004) and its application so far to vacuum Einstein gravity (Mallios, 2001; Mallios and Raptis, 2003; Mallios, 2005b). With these in hand, in the following section we present the category $\mathfrak{DT}_{fcq}$ of fintriads involved in our $fcqv$-perspective on Lorentzian QG (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004), which is a full subcategory of $\mathfrak{DT}$, as an ET in the original sense due to Lawvere and Tierney (MacLane and Moerdijk, 1992). Then, in Section 4 we present the same (fin)sheaf category as a GT by assuming a Grothendieck-type of stance against Sorkin's locally finite poset substitutes of continuous (i.e., $\mathcal{C}^0$-manifold) topologies. This generalization rests essentially on identifying certain covering coarse graining finsieves associated with Sorkin's locally finite open covers of the original (spacetime) continuum X and on observing that they define a Grothendieck topology on the poset category of open subsets of X. Under the categorical prism of ADG as developed by Papatriantafillou, an offshoot of the Grothendieck perspective on Sorkin is the categorical recasting of topological refinement in Sorkin's inverse systems of finitary poset substitutes of X in terms of differential geometric morphisms. This gives a differential geometric flavor to Sorkin's originally purely topological acts of refinement, while the finite, but more importantly the *infinite*, bicompleteness of the fintopos $\mathfrak{DT}_{fcq}$ secures the existence of a 'classical' continuum limit (Raptis and Zapatrin, 2000, 2001; Mallios and Raptis, 2002, 2003) (triad) of the coarse graining inverse system of fintriads in $\mathfrak{DT}_{fcq}$. In this respect, we observe that the abstract expression of the Principle of General Covariance (PGC) of GR as the functoriality of the ADG-vacuum Einstein gravitational dynamics with respect to the structure sheaf $\mathbf{A}$ of generalized coordinates is preserved under differential geometric refinement. The paper concludes with a fairly detailed, but largely heuristic and tentative, discussion of four possible paths we could take along current trends in 'categorical quantum gravity' in order to further develop our topos-theoretic scheme on $fcqv$-ADG-gravity. More notably in this epilogue, we anticipate the aforesaid sheaf cohomological quantum commutation relations, which may be regarded as being responsible for the geometrico-logical obstructions observed in $\mathfrak{D}\mathfrak{T}_{fcq}$ and its associated (:representation) Hilbert fintopos $\mathfrak{H}_{fcq}$. For the reader's convenience and expository completeness, we have relegated the formal definitions of an abstract elementary and an abstract Grothendieck topos to two appendices at the end.

2. MATHEMATICAL FORMALITIES: THE CATEGORY OF DIFFERENTIAL TRIADS AND ITS PROPERTIES

2.1. ADG Preliminaries: The Physico-Mathematical Versatility and Import of Differential Triads

The principal notion in ADG is that of a *differential triad* $\mathfrak{T}$. Let us briefly recall it, leaving more details to the original sources (Mallios, 1998a,b, 2005b).

We thus assume an in principle *arbitrary* topological space X, which serves as the base localization space for the sheaves to be involved in $\mathfrak{T}$. A differential triad then is thought of as consisting of the following three ingredients:

1. A sheaf $\mathbf{A}$ of unital, commutative and associative $\mathbb{K}$-algebras ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) on X called the *structure sheaf* of generalized arithmetics in the theory.[6]
2. A sheaf $\mathbf{\Omega}$ of $\mathbb{K}$-vector spaces over X, which is an $\mathbf{A}(U)$-module ($\forall$ open U in X).
3. A $\mathbb{K}$-linear and Leibnizian relative to $\mathbf{A}$ map (:sheaf morphism) ∂ between $\mathbf{A}$ and Ω,

$$\partial : \mathbf{A} \longrightarrow \Omega \tag{1}$$

   which is the archetypical paradigm of a (flat) $\mathbf{A}$-*connection* in ADG (Mallios, 1988, 1989).

In toto, a differential triad is represented by the triplet:

$$\mathfrak{T} := (\mathbf{A}_X, \partial, \Omega_X) \tag{2}$$

Or, omitting the base topological space X (as we shall often do in the sequel), $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega})$.

A couple of additional technical remarks on differential triads are due here for expository completeness:

- The constant sheaf $\mathbf{K}$ of scalars $\mathbb{K}$ is naturally injected into $\mathbf{A}$: $\mathbf{K} \overset{\subset}{\hookrightarrow} \mathbf{A}$.

[6] 'Coordinates' or 'coefficient functions' are synonyms to 'arithmetics'.

- In general, a *vector sheaf* $\mathcal{E}$ in ADG is defined as a locally free $\mathbf{A}$-module of finite rank n, by which it is meant that, locally in X (:$\forall$ open $U \subseteq X$), $\mathcal{E}$ is expressible as a finite power (or equivalently, a finite Whitney sum) of $\mathbf{A}$: $\mathcal{E}(U) \simeq (\mathbf{A}(U))^n \equiv \mathbf{A}^n(U)$, with n a positive integer called the *rank* of the sheaf, and $\mathcal{E}(U) \equiv \Gamma(U, \mathcal{E})$ the space of local sections of $\mathcal{E}$ over U. It is also assumed that such a vector sheaf $\mathcal{E}$ is the dual of the $\mathbf{A}$-module sheaf Ω appearing in the triad in (2), i.e., $\mathcal{E}^* = \mathbf{\Omega}(\equiv \mathbf{\Omega}^1) = \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{A})$.
- As it has been repeatedly highlighted in thorough investigations on various properties and in numerous (physical) applications of differential triads (Papatriantafillou, 2000, 2001, 2003a,b, 2004; Mallios and Raptis, 2002, 2003, 2004; Raptis, 2005), the latter generalize differential (:$\mathcal{C}^\infty$-smooth) manifolds, and, *in extenso*, ADG abstracts from and generalizes the usual differential geometry of smooth manifolds—i.e., the standard Differential Calculus on manifolds, which we have hitherto coined *Classical Differential Geometry* (CDG) (Mallios and Raptis, 2001, 2002, 2003, 2004; Raptis, 2005). Indeed, CDG may be thought of as a 'reduction' (i.e., a particular instance) of ADG, when one assumes $\mathcal{C}^\infty_X$—the usual sheaf of germs of smooth ($\mathbb{K}$-valued) functions on X—as structure sheaf in the theory. In this particular case, X is a smooth manifold M, while the $\mathbf{\Omega}$ involved in the corresponding 'classical' differential triad is the usual sheaf of germs of local differential 1-forms on (:cotangent to) M.[7] However, and this is the versatility of ADG, one need *not* restrict oneself to $\mathbf{A} \equiv \mathcal{C}^\infty_X$ hence also to the usual theory (CDG on manifolds). Instead, one can assume 'non-classical' structure sheaves that may appear to be 'exotic' (e.g., non-functional) or very 'pathological' (e.g., singular) from the 'classical' vantage of the featureless smooth continuum and the CDG it supports, provided of course that these algebra sheaves of generalized arithmetics furnish one with a differential operator ∂ with which one can set up a triad in the first place. Parenthetically, an example of the said 'exotic,' non-functional structure sheaves that have been used in numerous applications of ADG to gravity are sheaves of *differential incidence algebras of finitary posets* (Mallios and Raptis, 2001, 2002, 2003, 2004; Raptis, 2005).[8] At the same time, as very 'pathological,' 'ultra-singular' structure sheaves, one may regard sheaves of Rosinger's *differential algebras of*

[7] *In summa*, when $\mathbf{A} \equiv \mathcal{C}^\infty_X$, X is a differential manifold M, $\mathcal{E}$ is the tangent bundle TM of smooth vector fields on M, while $\mathbf{\Omega}$ the cotangent bundle T^*M of smooth 1-forms on M, which is the dual to TM. Note here that in the purely algebraic (:sheaf-theoretic) ADG, there are *a priori* no such central CDG-notions as *base manifold*, *(co)tangent space* (to it), *(co)tangent bundle etc.* ADG deals directly with the algebraic structure of the sheaves involved (:the algebraic relations between their sections), without recourse to (or dependence on) a background geometrical 'continuum space' (:manifold) for its differential geometric support. In this sense ADG is completely Calculus-free (Mallios, 1998a,b).

[8] They are also due to appear in the sequel.

non-linear generalized functions (:distributions), hosting singularities of all kinds densely in the underlying X. These too have so far been successfully applied to GR (Mallios and Rosinger, 1999, 2001; Mallios, 2001; Mallios and Rosinger, 2002; Mallios, 2003, 2005a; Mallios and Raptis, 2004; Raptis, 2005; Mallios, 2005b).

Connection, Curvature, Field and (vacuum) Einstein ADG-Gravity. Differential triads are versatile enough to support such key differential geometric concepts as *connection* and *curvature*. They can also accommodate central GR notions such as the (vacuum) *gravitational field* and the (vacuum) Einstein differential equations that it obeys. For expository completeness, but *en passant*, let us recall these notions from Mallios (1998a), Mallios (1998b), Mallios (2001), Mallios and Raptis (2001), Mallios and Raptis (2002), Mallios and Raptis (2003), Raptis (2005), Mallios and Raptis (2004):

A-connections: An **A**-connection $\mathcal{D}$ is a ('curved') generalization of the (flat) ∂ in (1) and its corresponding differential triad (2). It too is defined as a **K**-linear and Leibnizian sheaf morphism, as follows

$$\mathcal{D} :\ \mathcal{E} \longrightarrow \boldsymbol{\Omega}(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \boldsymbol{\Omega} \cong \Omega \otimes_{\mathbf{A}} \mathcal{E} \tag{3}$$

- **Curvature of an A-connection:** With $\mathcal{D}$ in hand, we can define its curvature $R(\mathcal{D})$ diagrammatically as follows

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\mathcal{D}"] \arrow[dr, "R \equiv \mathcal{D}^1 \circ \mathcal{D}"'] & & \boldsymbol{\Omega}^1(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \boldsymbol{\Omega}^1 \arrow[dl, "\mathcal{D}^1"] \\ & \boldsymbol{\Omega}^2(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \boldsymbol{\Omega}^2 & \end{tikzcd} \tag{4}$$

for a higher-order prolongation $\mathcal{D}^2$ of $\mathcal{D}(\equiv \mathcal{D}^1)$. One can then define the Ricci curvature $\mathcal{R}$, as well as its trace—the Ricci scalar $\mathcal{R}$. $R(\mathcal{D})$, unlike $\mathcal{D}$ which is only a constant sheaf **K**-morphism, is an **A**-morphism, *alias*, an $\otimes_{\mathbf{A}}$-tensor (with $\otimes_{\mathbf{A}}$ the usual homological tensor product functor).

- **ADG-field:** In ADG, the pair

$$(\mathcal{E}, \mathcal{D}) \tag{5}$$

namely, a connection $\mathcal{D}$ on a vector sheaf $\mathcal{E}$, is generically called a *field*. $\mathcal{E}$ is thought of as the *carrier space* of the connection, and $\mathcal{D}$ acts on its (local) sections. Note that there is no base (spacetime) manifold whatsoever supporting the ADG-field, so that the latter is a manifestly (i.e., by definition/construction) background manifold independent entity.

- **Vacuum Einstein equations:** The *vacuum ADG-gravitational field* is defined to be the field $(\mathcal{E}, \mathcal{D})$ whose connection part has a Ricci scalar curvature $R(\mathcal{D})$ satisfying the vacuum Einstein equations

$$\mathcal{R}(\mathcal{E}) = 0 \tag{6}$$

on the carrier sheaf $\mathcal{E}$. (6) can be derived from the variation of an Einstein-Hilbert action functional $\mathfrak{E}\mathfrak{H}$ on the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ of $\mathbf{A}$-connections $\mathcal{D}$ on $\mathcal{E}$. In (vacuum) ADG-gravity, the sole dynamical variable is the gravitational connection $\mathcal{D}$, thus the theory has been coined '*pure gauge theory*' and the formalism supporting it '*half-order formalism*' (Mallios and Raptis, 2003; Raptis, 2005; Mallios and Raptis, 2004).

Overall, applications to gravity (classical or quantum) aside for the moment, and in view of the categorical perspective that we wish to adopt in the present paper, perhaps the most important remark that can be made about differential triads is that *they form a category* $\mathfrak{D}\mathfrak{T}$, in which, as befits the aforementioned generalization of CDG by ADG, the category $\mathcal{M}an$ of differential manifolds is embedded (Papatriantafillou, 2000). Thus, in the next subsection we recall certain basic categorical features of $\mathfrak{D}\mathfrak{T}$ from Papatriantafillou (2000, 2001, 2003a,b, 2004), which will prove to be very useful in our topos-theoretic musings subsequently.

2.2. The Categorical Perspective on Differential Triads

As noted above, in the present subsection we draw material and results from Papatriantafillou's inspired work on the properties of $\mathfrak{D}\mathfrak{T}$, ultimately with an eye towards revealing its true topos-theoretic colors. Thus, below we itemize certain basic features of $\mathfrak{D}\mathfrak{T}$, with potential topos-theoretic significance to us as we shall see in the next section, as were originally exposed in Papatriantafillou (2000, 2001, 2003a,b, 2004). For more technical details, such as formal definitions, relevant proofs, etc., the reader can refer to those original papers.

However, before we discuss the properties of $\mathfrak{D}\mathfrak{T}$, we must first emphasize that it is indeed a category proper. *Objects* in $\mathfrak{D}\mathfrak{T}$ are differential triads, while *arrows* between them are *differential triad morphisms*. Let us recall briefly from Papatriantafillou (2000, 2003b, 2004) what the latter stand for.

Enter Geometric Morphisms. To discuss morphisms of differential triads, we first bring forth from MacLane and Moerdijk (1992) a pair of (covariant) adjoint functors between sheaf categories that are going to be of great import in the sequel.

Let X, Y be topological spaces, and $\mathcal{S}\mathbf{hv}_X$, $\mathcal{S}\mathbf{hv}_Y$ sheaf categories over them. Then, a continuous map $f : X \longrightarrow Y$ induces a pair $\mathcal{GM}_f = (f_*, f^*)$ of (covariant) adjoint functors between $\mathcal{S}\mathbf{hv}_X$ and $\mathcal{S}\mathbf{hv}_Y$ ($f_* : \mathcal{S}\mathbf{hv}_X \longrightarrow \mathcal{S}\mathbf{hv}_Y$, $f^* : \mathcal{S}\mathbf{hv}_Y \longrightarrow \mathcal{S}\mathbf{hv}_X$) called *push-out* (:direct image) and *pull-back* (:inverse image), respectively. In topos-theoretic parlance, such a pair of adjoint functors is known as a *geometric morphism* (MacLane and Moerdijk, 1992).

With $\mathcal{GM}$ in hand, we are in a position to define differential triad morphisms. Let $\mathfrak{T}_X = (\mathbf{A}_X, \partial_X, \mathbf{\Omega}_X)$ and $\mathfrak{T}_Y = (\mathbf{A}_Y, \partial_Y, \mathbf{\Omega}_Y)$ be differential triads over the aforesaid topological spaces. Then, like the triads themselves, a *morphism* $\mathcal{F}$ between them is a triplet of maps $\mathcal{F} = (f, f_A, f_\Omega)$ having the following four properties relative to $\mathcal{GM}_f$:

1. the map $f : X \longrightarrow Y$ is continuous, as set by $\mathcal{GM}_f$;
2. the map $f_{\mathbf{A}} : \mathbf{A}_Y \longrightarrow f_*(\mathbf{A}_X)$ is a morphism of sheaves of $\mathbb{K}$-algebras over Y, which preserves the respective algebras' unit elements (i.e., $f_{\mathbf{A}}(1) = 1$);
3. the map $f_\Omega : \mathbf{\Omega}_Y \longrightarrow f_*(\mathbf{\Omega}_X)$ is a morphism of sheaves of $\mathbb{K}$-vector spaces over Y, with $f_\Omega(\alpha\omega) = f_{\mathbf{A}}(\alpha) f_{\mathbf{\Omega}}(\omega),\ \forall(\alpha, \omega) \in \mathbf{A}_Y \times_Y \mathbf{\Omega}_Y$; and finally,
4. with respect to the $\mathbf{K}$-linear, Leibnizian sheaf morphism ∂ in the respective triads, the following diagram is commutative:

$$\begin{tikzcd} \mathbf{A}_Y \arrow[r, "\partial_Y"] \arrow[d, "f_{\mathbf{A}}"'] & \mathbf{\Omega}_Y \arrow[d, "f_{\mathbf{\Omega}}"] \\ f_*(\mathbf{A}_X) \arrow[r, "f_*(\partial_X)"'] & f_*(\mathbf{\Omega}_X) \end{tikzcd}$$

reading: $f_\Omega \circ \partial_Y = f_*(\partial_X) \circ f_{\mathbf{A}}$.

To complete the argument that $\mathfrak{DT}$ is a true category, we note that for each triad $\mathfrak{T}_X$ there is an *identity morphism* $id_{\mathfrak{T}_X} := (id_x, id_{\mathbf{A}}, id_{\mathbf{\Omega}})$ defined by the corresponding identity maps of the spaces involved in the triad. There is also an *associative composition law* (:product) between triad morphisms (Papatriantafillou, 2000, 2003b, 2004), making thus $\mathfrak{DT}$ an arrow (:triad morphism) semigroup, complete with identities (:units)—one for every triad object in it.

Here, we would like to make some auxiliary and clarifying remarks about $\mathfrak{DT}$ that will prove to be helpful subsequently:

- For a given base space X, the collection $\{(\mathfrak{T}_X, \mathcal{F}_X := (id_X, f_{\mathbf{A}}, f_{\mathbf{\Omega}}))\}$ constitutes a subcategory of $\mathfrak{DT}$, symbolized as $\mathfrak{DT}_X$.
- In general, differential triad morphisms are thought of as *maps that preserve the purely algebraic (:sheaf-theoretic) differential (geometric) structure or 'mechanism' encoded in every triad*. They are *abstract differentiable maps*, generalizing in many ways the usual smooth ones between differential manifolds in $\mathcal{M}an$.

- Indeed, following the classical (CDG) jargon, in ADG we say that a continuous map $f: X \longrightarrow Y$ is *differentiable*, if it can be completed to a differential triad morphism. Such continuity-to-differentiability completions of maps, in striking contradistinction to the case of $\mathcal{M}an$ and CDG, are always feasible in $\mathfrak{DT}$ and ADG, as the following two results show:
- If X and Y are topological spaces as before, $f: X \longrightarrow Y$ continuous, and X carries a differential triad $\mathfrak{T}_X$, the push-out f_* induces a differential triad on Y. Vice versa, if Y carries a differential triad, the pull-back f^* endows X with a differential triad. Furthermore, these so-called '*final and initial differential structures*' respectively, satisfy certain *universal mapping* relations that promote f to a *differentiable* map in the sense above—i.e., they complete it to a triad morphism (Papatriantafillou, 2003a,b, 2004).[9] This is in glaring contrast with the usual situation in $\mathcal{M}an$, whereby if X is a smooth manifold equipped with an atlas $\mathcal{A}$, while Y is just a topological space, one cannot push-forward $\mathcal{A}$ by f_* in order to make Y a differential manifold and in the process turn f into a smooth map. Similarly for the reverse scenario in which Y is a differential manifold charted by a smooth atlas $\mathcal{B}$, and X simply a topological space: f^* cannot 'pull back' $\mathcal{B}$ on X thus promote the latter into a smooth space.[10]
- Much in the same fashion, if X is a differential manifold and $\sim$ an (arbitrary) equivalence relation on it, the moduli space $X/\sim$ does not inherit, via the (possibly continuous) canonical projection $f_\sim: X \longrightarrow X/\sim$, the usual differential structure of X (and, accordingly, the possibly continuous surjection f does *not* become differentiable in the process). This is not the case in $\mathfrak{DT}$; whereby, when the base space of a differential triad is modded-out by an equivalence relation, the resulting quotient space inherits the original triad's structure (i.e., it becomes itself a differential triad), and in the process $f_\sim$ becomes a triad morphism (Papatriantafillou, 2003a,b, 2004). This particular example of the versatility of $\mathfrak{DT}$ (and ADG!), as contrasted against the 'rigidity' of $\mathcal{M}an$ (and CDG!), has been exploited numerous times in the past, especially in the finitary case of Sorkin (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004). We shall exploit it again later in this paper.

[9] Furthermore, as shown in Papatriantafillou (2003a,b, 2004), the composition of a differentiable map with a continuous one also becomes differentiable in the sense above.

[10] Results like this have prompted workers in ADG to develop, as an extension of the usual '*differential geometry of smooth manifolds*' (:CDG in $\mathcal{M}an$), what one one could coin '*the differential geometry of topological spaces*' (ADG in $\mathfrak{DT}$). This possibility hinges on the following 'Calculus-reversal' observed in ADG and noted above: as in the usual CDG on manifolds '*differentiability implies continuity*' (:a smooth map is automatically continuous), in ADG the converse is also possible; namely, that '*continuity implies differentiability!*' (:a continuous map can become differentiable).

After these telling preliminaries, we return to discuss the categorical properties of $\mathfrak{DT}$ that will be of potential topos-theoretic significance in the sequel. Once again, we itemize them, commenting briefly on every item:

- $\mathfrak{DT}$ *is bicomplete*. This means that $\mathfrak{DT}$ is *closed* under both *inverse* (*alias*, projective) and *direct* (*alias*, inductive) limits of differential triads (Papatriantafillou, 2001, 2004). In particular, it is closed under *finite* limits (projective) and colimits (inductive)—i.e., it is *finitely bicomplete*.[11]
- $\mathfrak{DT}$ *has canonical subobjects*. As it has been shown in detail in Papatriantafillou (2003b, 2000, 2004), "*every subset of the base space of a differential triad defines a differential triad, which is a subobject of the former*."[12] On the other hand, $\mathcal{M}an$ manifestly lacks this property, since it is plain that an arbitrary subset of a differential manifold is not itself a manifold.
- $\mathfrak{DT}$ *has finite products*. In Papatriantafillou (2003b, 2000, 2004) it is also shown that there are *finite cartesian products* of differential triads in $\mathfrak{DT}$.
- $\mathfrak{DT}$ *has an exponential structure*. This means that given any two differential triads $\mathfrak{T}, \mathfrak{T}' \in \mathfrak{DT}$, one can form the collection $\mathfrak{T}'^{\mathfrak{T}}$ of all triad morphisms in $\mathfrak{DT}$ from $\mathfrak{T}$ to $\mathfrak{T}'$. Common in categorical notation is the alternative designation of $\mathfrak{T}'^{\mathfrak{T}}$ by $\mathcal{H}om(\mathfrak{T}, \mathfrak{T}')$ (:'hom-sets of triad morphisms'). In addition, the exponential is supposed to effectuate canonical isomorphisms relative to the aforementioned (cartesian) product in $\mathfrak{DT}$. Thus, for $\mathfrak{T}, \mathfrak{T}'$ as above and $\mathfrak{T}''$ any other triad in $\mathfrak{C}$: $\mathcal{H}om(\mathfrak{T}'' \times \mathfrak{T}, \mathfrak{T}') \simeq \mathcal{H}om(\mathfrak{T}'', \mathfrak{T}'^{\mathfrak{T}})$ (or equivalently: $\mathfrak{T}'^{\mathfrak{T}'' \times \mathfrak{T}} \simeq (\mathfrak{T}'^{\mathfrak{T}})^{\mathfrak{T}''}$).

3. THE CATEGORY OF FINTRIADS AS AN ET

The high-point of the present section is that we show that the category $\mathfrak{DT}_{fcq}$ of fintriads, which is a subcategory of $\mathfrak{DT}$, is an ET (MacLane and Moerdijk, 1992). Before we do this however, let us first recall *en passant* the finitary perspective on continuous (spacetime) topology as originally championed by Sorkin (1991) and then extended by the sheaf-theoretic ADG-means to the differential geometric realm, with numerous physical applications to discrete and quantum spacetime structure (:causets and qausets) (Raptis and Zapatrin, 2000, 2001; Raptis, 2000a,b), vacuum Einstein-Lorentzian gravity (classical and quantum) (Mallios and Raptis, 2001, 2002, 2003), and gravitational singularities (Raptis, 2005; Mallios and Raptis, 2004).

[11] The synonyms 'co-complete' or 'co-closed' are often used instead of 'bicomplete.'
[12] Excerpt from the abstract of Papatriantafillou (2003b).

3.1. 'Finatarities' Revisited

Below, we give a short, step-by-step historical anadromy to the development of the finitary spacetime and gravity program by ADG-theoretic means, isolating and highlighting the points that are going to be relevant to our ADG *cum* topos-theoretic efforts subsequently.[13] The account concludes with the arrival at the category $\mathfrak{D}\mathfrak{T}_{fcq}$ of fintriads, which we present as an ET proper in the following two subsections:

- **Finitary poset substitutes of topological manifolds.** In Sorkin (1991), Sorkin commenced the finitary spacetime program solely with topology in mind. Namely, he substituted the usual continuous ($\mathcal{C}^0$) topology of an open and bounded region X of the spacetime manifold M by a poset P_i relative to a locally finite open covering $\mathcal{U}_i$[14] of X. He arrived at P_i, which is a T_0-topological space in its own right, by factoring out X by the following *equivalence relation* between X's points:

$$x \overset{\mathcal{U}_i}{\sim} y \Leftrightarrow \Lambda(x)|_{\mathcal{U}_i} = \Lambda(y)|_{\mathcal{U}_i},\ \forall x, y \in X; \qquad P_i := X/\overset{\mathcal{U}_i}{\sim} \tag{7}$$

  with $\Lambda(x)|_{\mathcal{U}_i} := \cap\{U \in \mathcal{U}_i : x \in U\}$ the smallest open set in $\mathcal{U}_i$ (or equivalently, in the subtopology τ_i of X generated by taking arbitrary unions of finite intersections of the covering open sets in $\mathcal{U}_i$) containing x. $\Lambda(x)|_{\mathcal{U}_i}$ is otherwise known as the Alexandrov-Čech *nerve* of x relative to the open covering $\mathcal{U}_i$ (Raptis and Zapatrin, 2000). The 'points' of P_i are equivalence classes (:nerves) of X's points, partially ordered by set-theoretic inclusion '$\subseteq$,' with the said equivalence relation being interpreted as 'indistinguishability' of points relative to our 'coarse measurements' in $\mathcal{U}_i$. That is to say, two points (:'events') of X in the same class cannot be distinguished (or 'separated,' topologically speaking) by the covering open sets (:our 'coarse measurements') in $\mathcal{U}_i$.

  Sorkin's scenario for approximating (:'substituting') the locally Euclidean (:continuum) topology of X by the finitary topological posets (:fintoposets) P_i hinges on the fact that the latter constitute an *inverse* (*alias*, *projective*) system $\overleftarrow{\mathcal{P}}$ relative to a *topological refinement net* $I \equiv \overleftarrow{\mathfrak{U}}_i := \{(\mathcal{U}_i, \preceq\}_{i\in I}$ of the fincovers of X. Here, the partial order $\mathcal{U}_i \preceq \mathcal{U}_j$ is interpreted as follows: '*the covering $\mathcal{U}_j$ (resp. $\mathcal{U}_i$) is finer (resp. coarser) than $\mathcal{U}_i$ (resp. $\mathcal{U}_j$).*' Equivalently, $\mathcal{U}_j$ is a refinement of $\mathcal{U}_i$, and thus the latter is a subcover of the former. Correspondingly, τ_i is a subtopology of τ_j. Henceforth, the net I will be treated as an index-set labelling the

[13] For more details on what follows, the reader is referred to the aforementioned papers on the finitary ADG-based approach to spacetime and gravity.

[14] The index 'i' will be explained shortly.

open coverings of X and the corresponding fintoposets. However, we may use the symbols $\overleftarrow{\mathfrak{U}}_i$ and I interchangeably, hopefully without causing any confusion. Parenthetically, and with an eye towards our subsequent topos-theoretic labors in the light of $\mathfrak{D}\mathfrak{T}$, it is worth pointing out here that $\overleftarrow{\mathcal{P}}$ can be described as an I-indexed family of pentads:

$$\overleftarrow{\mathcal{P}} := \{(X, f_i, P_i, f_j, P_j, f_{ji})\}, \; (i \preceq j \in I) \tag{8}$$

whereby, F_i (resp. F_j) is a continuous surjection (:projection map) from X to P_i (resp. P_j), while f_{ji} a continuous fintoposet morphism from P_j to P_i corresponding to the act of topological refinement when one refines the open cover $\mathcal{U}_i$ to $\mathcal{U}_j$. *En passant*, we note that the epithet 'continuous' for f_{ji} above pertains to the fact that one can assign a 'natural' topology—the so-called Sorkin *lower-set* or *sieve-topology*—to the P_is, whereby an open set is of the form $\mathcal{O}(x) := \{y \in P_i : \; y \longrightarrow x\}$, with '$\longrightarrow$' the partial order relation in P_i. Basic open sets for the Sorkin topology are defined via the *links* or *covering* ('immediate arrow') relations in (the Hasse diagram of) P_i: $\mathcal{O}_B(x) := \{y \in P_i : \; (y \longrightarrow x) \wedge (\nexists z \in P_i : \; y \longrightarrow z \longrightarrow x)\}$. Then, f_{ji} is a monotone (:partial order preserving) surjection from P_j to P_i, hence it is continuous with respect to the said Sorkin sieve-topology. Accordingly, the arrow $x \longrightarrow y$ can be literally interpreted as the convergence of the constant sequence (x) to y in the Sorkin topology (Sorkin, 1991). This sieve-topology of Sorkin will prove to be very important for our topos-theoretic (and especially the GT) musings in the sequel.[15]

From Sorkin (1991) we also recall that there is a *universal mapping condition* obeyed by the triplets (F_i, F_j, f_{ji}) of continuous surjective maps in $\overleftarrow{\mathcal{P}}$, which looks diagrammatically as follows

$$\begin{tikzcd} X \arrow[r, "F_j"] \arrow[dr, "F_i"'] & P_j \arrow[d, "f_{ji}"] \\ & P_i \end{tikzcd} \tag{9}$$

and reads: $F_i = f_{ji} \circ F_j$. That is, the system $(F_i)_{i \in I}$ of canonical projections of X onto the fintoposets is '*universal*' as far as maps between T_0-spaces are concerned, with f_{ji} the *unique* map—itself a partial order preserving (monotone) surjection of P_j onto P_i—mediating between the continuous projections F_i and F_j of X onto the T_0-fintoposets P_i and P_j, respectively.

[15] See next section and Appendix B.

Then, the central result in Sorkin (1991)—the one that qualifies the fintoposets as genuine finitary approximations of the continuous topology of the $\mathcal{C}^0$-manifold X—is that, thanks to the universal mapping property that the P_is enjoy, at the projective limit of infinite topological refinement (:coarse graining) of the underlying coverings in I, $\overleftarrow{\mathcal{P}}$ effectively[16] yields back the original topological continuum X (up to homeomorphism). Formally, one writes:

$$\lim_{\infty \leftarrow j} f_{ji}(P_i) =: P_\infty \underset{\text{homeo.}}{\overset{F_\infty}{\leftrightarrows}} X \text{ (modulo Hausdorff reflection)} \tag{10}$$

Let it be noted here that this universal mapping property of the maps between the T_0-fintoposets above is completely analogous to the one possessed by the differential triad morphisms (e.g., the push-outs and pull-backs along continuous maps between the triads' base topological spaces) (Papatriantafillou, 2003a,b, 2004) mentioned earlier in 2.2. In fact, shortly, when we discuss fintriads and their inverse limits, the ideas of Sorkin and Papatriantafillou will appear to be tailor-cut for each other; albeit, with the ADG-based work of Papatriantafillou adding an important *differential* geometric slant to Sorkin's originally purely topological considerations.

- **Incidence algebras of T_0-posets.** One can use a discrete version of Gel'fand duality to represent the fintoposets P_i above algebraically, as so-called *incidence algebras* (write $\Omega_i(P_i)$, and read '*the incidence algebra* Ω_i *of the fintoposet* P_i') (Raptis and Zapatrin, 2000, 2001). The correspondence $P_i \longrightarrow \Omega_i$ is manifestly *functorial*,[17] especially when one regards the P_is as graded simplicial complexes having for simplices the aforementioned Čech-Alexandrov nerves (Raptis and Zapatrin, 2000, 2001; Zapatrin, 2001a).

  The Ω_is, being (categorically) dual objects to the 'discrete' homological (:simplicial) P_is, may be regarded as '$\mathbb{Z}_+$-*graded discrete differential* $\mathbb{K}$-*algebras*'—reticular cohomological analogues of the usual spaces (:modules) of (smooth) differential forms on the manifold X in focus (Raptis and Zapatrin, 2000, 2001). Indeed, they were seen to be '*discrete differential manifolds*' (in the sense of Dimakis and Müller-Hoissen (1994); Dimakis *et al.* (1995); Dimakis and Müller-Hoissen (1999)), as follows

$$\Omega_i = \bigoplus_{p\in\mathbb{Z}_+} \Omega_i^p = \overbrace{\Omega_i^0}^{\mathcal{A}_i} \oplus \overbrace{\Omega_i^1 \oplus \Omega_i^2 \oplus \ldots}^{\mathcal{R}_i} \equiv \mathcal{A}_i \oplus \mathcal{R}_i \tag{11}$$

[16] That is, modulo Hausdorff reflection (Kopperman and Wilson, 1997).

[17] The reverse correspondence $\Omega_i \longrightarrow P_i$ having been coined *Gel'fand spatialization* (Zapatrin, 1998; Raptis and Zapatrin, 2000, 2001).

where $\mathcal{R}_i$ is a $\mathbb{Z}_+$-graded $\mathcal{A}_i$-bimodule of (exterior, real or complex) differential form-like entities Ω_i^p ($p \geq 1$),[18] related within each Ω_i by nilpotent Cartan-Kähler-type of (exterior) differential operators $d_i^p : \Omega_i^p \longrightarrow \Omega_i^{p+1}$, with $d_i^0 \equiv \partial_i : \Omega^0 \longrightarrow \Omega^1$ the finitary analogue of the standard derivation ∂ in (2) and $d_i^p : \Omega_i^p \longrightarrow \Omega_i^{p+1}$ ($p \geq 1$) its higher order (grade or degree) prolongations.

Plainly, it is tacitly assumed here that the locally Euclidean X, apart from the continuous ($\mathcal{C}^0$) topology, also carries the usual differential (:$\mathcal{C}^\infty$-smooth) structure, which in turn the Ω_is can be thought of as approximating 'discretely' ('finitarily'). Thus, the cohomological Ω_is too are seen to comprise an inverse system $\overleftarrow{\Omega}$ relative to the aforesaid topological refinement net of fincovers of X (Mallios and Raptis, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004).

- **Finsheaves of incidence algebras: fintriads.** The key observation in arriving at *finsheaves* (Raptis, 2000b) of incidence algebras is that by construction (i.e., by the aforesaid method of Gel'fand spatialization) the map $\Omega_i \longrightarrow P_i$ is a *local homeomorphism*, *alias*, a *sheaf* (MacLane and Moerdijk, 1992; Mallios, 1998a,b). Thus, finsheaves $\mathbf{\Omega}_i(P_i)$ of incidence algebras over Sorkin's fintoposets were born (Mallios and Raptis, 2001). Moreover, since the Ω_is carry not only topological, but also *differential geometric* structure as noted above, the $\mathbf{\Omega}_i$s may be thought of as finitary analogues ('approximations') of the 'classical' (:$\mathcal{C}^\infty$-smooth) differential triad $\mathfrak{T}_\infty$ supported by the differential manifold X. They are coined '*fintriads*' and they are fittingly symbolized by $\mathfrak{T}_i$ (Mallios and Raptis, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004).

  There are actually two ways to arrive at $\mathfrak{T}_i$s—one 'indirect' and 'constructive,' the other 'direct' and 'inductive':

  1. The 'indirect-constructive' way is the one briefly described above, namely, by first obtaining the P_is from Sorkin's factorization algorithm, then by defining the corresponding Ω_is and suitably topologizing them in a 'discrete' Gel'fand representation (:duality) fashion, and finally, by defining finsheaves of the latter (regarded as discrete differential algebras) over the former. This is the path we followed originally in our work (Mallios and Raptis, 2001, 2002, 2003).
  2. The 'direct-inductive' way goes as follows (Raptis, 2005; Mallios and Raptis, 2004): one simply starts with the 'classical' smooth

[18] In (11) above, $\mathcal{A}_i \equiv \Omega_i^0$ is a commutative subalgebra of Ω_i called *the algebra of coordinate functions in* Ω_i, while $\mathcal{R}_i \equiv \bigoplus_i^{p\geq 1} \Omega_i^p$ a linear subspace of Ω_i called *the module of differentials over* $\mathcal{A}_i$. The elements of each linear subspace Ω_i^p of Ω_i in $\mathcal{R}_i$ have been regarded as 'discrete' analogues of the usual smooth differential p-forms teeming the usual (cotangent bundle over the) smooth manifold (Raptis and Zapatrin, 2000, 2001).

differential triad $\mathfrak{T}_\infty$ on the (differential) manifold X and, by calling forth Papatriantafillou's push-out/pull-back results in Papatriantafillou (2003a, 2004) that we mentioned back in 2.2, one induces the usual differential structure, via the push-out F_{i*} of the continuous surjection $F_i : X \longrightarrow P_i$ in (9) above, from X to the $\overset{\mathcal{U}_i}{\sim}$-moduli space P_i. In the process, F_i becomes differentiable— ie, it lifts to a triad morphism $\mathcal{F}_i : \mathfrak{T}_\infty \longrightarrow \mathfrak{T}_i$. Incidentally, from Papatriantafillou's results (Papatriantafillou, 2003a,b, 2004) it follows that the continuous surjection f_{ji} in (9) is also promoted to a fintriad morphism $\mathcal{F}_{ji} : \mathfrak{T}_j \longrightarrow \mathfrak{T}_i$.

From Mallios and Raptis (2001, 2002, 2003); Raptis (2005); Mallios and Raptis (2004) we draw that a fintriad can be symbolized as $\mathfrak{T}_i = (\mathbf{A}_i, \partial_i, \mathbf{\Omega}_i)$, where $\mathbf{A}_i$ is a unital, abelian, associative algebra (structure) sheaf whose stalks are inhabited by elements (:coordinate function-like entities) of $\mathcal{A}_i$ in (11), while $\mathbf{\Omega}_i$ is an $\mathbf{A}_i$-bimodule with elements (:differential form-like entities) of $\mathcal{R}_i$ in (11) dwelling in its fibers.

- **The category $\mathfrak{DT}_{fcq}$ and its completeness in $\mathfrak{DT}$.** Finally, we can organize the $\mathfrak{T}_i$s into the category $\mathfrak{DT}_{fcq}$ of fintriads. Objects in $\mathfrak{DT}_{fcq}$ are the said fintriads, while arrows between them fintriad morphisms. It is easy to see that $\mathfrak{DT}_{fcq}$ is a *full subcategory* of $\mathfrak{DT}$ (MacLane and Moerdijk, 1992).

  An important result about $\mathfrak{DT}_{fcq}$ is that it is finitely complete in itself, and infinitely complete in $\mathfrak{DT}$— ie, it is closed under finite projective limits, and closed in $\mathfrak{DT}$ under infinite inverse limits. This is a corollary result which derives—also bearing in mind Sorkin's inverse limit result about $\overleftarrow{\mathcal{P}}$ (10), as well as the universal mapping properties observed in both Sorkin's scheme (Sorkin, 1991) and in $\mathfrak{DT}$ (Papatriantafillou, 2003a,b, 2004)—from the following theorem proved by Papatriantafillou in Papatriantafillou (2001, 2004):[19]

  Theorem. Let $[\mathfrak{T}_i = (\mathbf{A}_i, \partial_i, \mathbf{\Omega}_i); \mathcal{F}_{ji} = (f_{ji}, f_{ji\mathbf{A}}, f_{ji\mathbf{\Omega}})]_{i \preceq j \in I}$ be a projective system in $\mathfrak{DT}_{fcq} \subset \mathfrak{DT}$ and let P_i be the base space of each $\mathfrak{T}_i$, with $\overleftarrow{\mathcal{P}} = (P_i, f_{ji})$ their inverse system considered above. There is a differential triad $\mathfrak{T}_\infty$ over the inverse limit space $X \overset{\text{homeo.}}{\simeq} P_\infty$ in (10), satisfying the universal property of the projective limit in $\mathfrak{DT}$.

  Dually, the same would hold for an *inductive system* of differential triads over an inductive system of base spaces and their direct limit space (Papatriantafillou, 2001, 2004). Here, in connection with Sorkin's 'finitarities' (Sorkin, 1991), we happen to be interested only in the projective

[19] Quoting the author almost verbatim from Papatriantafillou (2001) (Theorem 4.4), with slight changes in notation and language to suit our finitary considerations here.

(inverse) limit case, but $\mathfrak{D}\mathfrak{T}_{fcq}$ is also co-complete in $\mathfrak{D}\mathfrak{T}$. In fact, it is noteworthy here that *inductive* systems of fintoposets (as base spaces) were originally employed in Raptis (2000b) in order to define finsheaves as finitary approximations of the sheaf $\mathcal{C}^0_X$ of continuous functions over the topological manifold X. Indeed, the stalks of the latter, which host the germs of continuous functions on X, were seen to arise as inductive limits of the said finsheaves at the limit of infinite topological refinement of the underlying open covers $\mathcal{U}_i$.

- **fcqv-ADG-gravity.** Parenthetically, in closing this subsection, we must note the significant physical import of fintriads in ADG-gravity. So far, we have been able to formulate a *manifestly background differential spacetime manifold independent vacuum Einstein-Lorentzian gravity as a pure gauge theory*,[20] with finitistic, causal and quantum traits built into the theory from the very beginning (Mallios and Raptis, 2001, 2002, 2003). The high-point in those investigations is that every fintriad $\mathfrak{T}_i$, equipped with a finitary connection $\mathcal{D}_i$ (:a finitary instance of (3)) and its associated curvature R_i (:a finitary example of (4)), is seen to support a finitary version of the vacuum Einstein equations. (6):

$$\mathcal{R}_i(\mathcal{E}_i) = 0 \tag{12}$$

  with geometric prequantization traits already attributed to $\mathcal{E}_i$ (e.g., its local sections have been sheaf cohomologically interpreted as quantum particle states of the 'field of quantum causality'—fittingly called 'causons' (Mallios and Raptis, 2002)).

  Moreover, we have made thorough investigations on how ADG-gravity can evade singularities of the most pathological kind,[21] and their associated unphysical infinities (Mallios and Rosinger, 1999, 2001; Mallios, 2001; Mallios and Rosinger, 2002; Mallios, 2002, 2003, 2005a, 2004) (especially (Mallios and Raptis, 2004)), with a special application to the finitary-algebraic 'resolution' of the inner Schwarzschild singularity of the gravitational field of a point-particle (Raptis, 2005).

The $fcqv$-ADG-gravitational dynamics (12) may be thought of as 'taking place' within the category $\mathfrak{D}\mathfrak{T}_{fcq}$, which in turn may be regarded as a mathematical 'universe' ('space') of (dynamically) varying qausets. We shall return to comment more on this in 4.3 and subsequent sections, after we show that $\mathfrak{D}\mathfrak{T}_{fcq}$ is actually a finitary example of an EGT—a fintopos. The crux of the argument here is

[20] That is, a formulation of gravity solely in terms of an algebraic **A**-connection field $\mathcal{D}$.

[21] Like for instance when Rosinger's differential algebras of generalized functions (:non-linear distributions), hosting singularities everywhere densely in the background topological space(time) X, are used as structure sheaves in the theory (:'*spacetime foam differential triads*') (Mallios and Rosinger, 2001, 2002).

that as every sheaf (and *in extenso* topos) of, say, sets, can be thought of as a (mathematical) world of varying sets (MacLane and Moerdijk, 1992), so $\mathfrak{D}\mathfrak{T}_{fcq}$ may be thought of as a universe of dynamically variable (qau)sets, varying under the influence (action) of the $fcqv$-ADG-gravitational field $\mathcal{D}_i$. We thus first turn to the topos-theoretic perspective on $\mathfrak{D}\mathfrak{T}_{fcq}$ next.

3.2. $\mathfrak{D}\mathfrak{T}_{\mathbf{fcq}}$ as a Finitary Example of an ET

The title of the present subsection is one of the two main mathematical results in the present paper—the other being that $\mathfrak{D}\mathfrak{T}_{fcq}$ is also a finitary instance of a GT-like structure, as we shall show in the next section.

First, let us stress the following subtle point: $\mathfrak{D}\mathfrak{T}_{fcq}$ is a category of (fin)sheaves *not* over a fixed topological space like the usual sheaf categories (:topoi) encountered in standard topos theory (MacLane and Moerdijk, 1992), but over '*variable*' finitary topological spaces (:fintoposets)—spaces that 'vary' with topological refinement (:coarse graining) and the associated 'degree $i \in I$ of topological resolution,' as described above.[22]

Now, the arrival at the result that $\mathfrak{D}\mathfrak{T}_{fcq}$ is an ET (:a cartesian closed category) is quite straightforward: one simply has to juxtapose the properties of $\mathfrak{D}\mathfrak{T}$, as they were gathered from Papatriantafillou (2000, 2001, 2003a,b, 2004) and presented at the end of Subsection 2.2, against the formal definitional properties of an ET *à la* Lawvere and Tierney, as taken from MacLane and Moerdijk (1992) and synoptically laid out in appendix A at the end. Then, one should bring forth that $\mathfrak{D}\mathfrak{T}_{fcq}$ is a full subcategory of $\mathfrak{D}\mathfrak{T}$, enjoying all the latter's formal properties. Thus, to recapitulate these properties, $\mathfrak{D}\mathfrak{T}_{fcq}$ is an ET because it:

- is closed under finite limits and colimits (:it is finitely bicomplete); moreover, it is closed even 'asymptotically' (ie, under infinite topological refinement of the base P_is and their underlying $\mathcal{U}_i$s) in $\mathfrak{D}\mathfrak{T}$, as we saw earlier;
- has finite (cartesian) products and coproducts (direct sums);
- has an exponential structure given, for any pair of fintriads, by continuous maps f_{ji} between the underlying fintoposets and the fintriad morphisms that these maps lift to; moreover, this structure 'intertwines' canonically with the said cartesian product as explained above; and finally,
- has canonical subobjects. That is to say, it has a subobject classifier that it inherits naturally from the archetypical topos $\mathcal{S}\mathbf{hv}(X)$ of sheaves (of structureless sets, for example; or for instance, from the topos $\mathcal{S}\mathbf{hv}^0(X)$ of sheaves $\mathcal{C}^0_X$ of rings of continuous $\mathbb{K}$-valued functions) over the original

[22] This remark will prove to be important in the sequel when we interpret $\mathfrak{D}\mathfrak{T}_{fcq}$ as a finitary replacement of the classical 'continuum topos' $\mathcal{S}\mathbf{hv}^0(X)$, and in the last section, where we shall remark on the possibility of relating our scheme to Isham's 'quantizing on a category' scenario.

topological manifold X. This will become more transparent in the next section where we present $\mathfrak{D}\mathfrak{T}_{fcq}$ alternatively as a GT.

An important question that arises from the exposition above is the following: *what is the subobject classifier Ω in $\mathfrak{D}\mathfrak{T}_{fcq}$, and perhaps more importantly, what is its physical interpretation?* For this, the reader will have to wait for our ADG-based Grothendieck-type of perspective on Sorkin's finitary scheme in the next section.

4. AN ADG-THEORETIC GROTHENDIECK-TYPE OF PERSPECTIVE ON SORKIN IN $\mathfrak{D}\mathfrak{T}_{fcq}$

In this section we give an alternative topos-theoretic description of the ET $\mathfrak{D}\mathfrak{T}_{fcq}$. We present it as a finitary example of a GT-like structure. In a way, a Grothendieck-type of perspective on the finitary topology scenario of Sorkin (1991) is perhaps more 'canonical' and 'natural' than the (more abstract) ET vantage for viewing $\mathfrak{D}\mathfrak{T}_{fcq}$ as a topos proper, because in Sorkin's work, as we witnessed in the previous section, such notions as *covering*, *sieve-topology* and its associated *topological coarse graining* procedure, figure prominently in the theory and they have well known, direct and generalized correspondents in Grothendieck's celebrated work (MacLane and Moerdijk, 1992).

At the same time, by presenting $\mathfrak{D}\mathfrak{T}_{fcq}$ as a type of GT will enable us to see straightforwardly what the subobject classifier Ω_{fcq} is in it. With Ω_{fcq} in hand, and by viewing it as a '*generalized truth values object*' as it is its customary logical interpretation in standard topos theory (MacLane and Moerdijk, 1992), we shall then open paths for potentially exploring deep connections between (spacetime) geometry (e.g., topology) and (quantum) logic. In fact, since $\mathfrak{D}\mathfrak{T}_{fcq}$ carries *differential* geometric (not just topological) information, and since in the past we have successfully employed this structure to model $fcqv$-ADG-gravity, the road will be open for investigating close relationships between the differential geometric ('gravitational') structure of the world, and its quantum logical traits. We shall explore two such potential relationships with important physical interpretation and implications in the epilogue. In the present section however, we just present the topological refinement in Sorkin's scheme by *differential geometric morphisms* in $\mathfrak{D}\mathfrak{T}_{fcq}$.

Ex altis viewed, and more from a technical (:mathematical) vantage, invoking Grothendieck's categorical ideas in an ADG-theoretic context appears to be only natural, since the machinery of (abstract) *sheaf cohomology* is central in the ADG-technology (Mallios, 1998a,b, 2005b), while (abstract) *sheaf cohomology* was originally the *raison d'être et de faire* of Grothendieck's pioneering category-theoretic work in general homological algebra. For it is no exaggeration to say that Grothendieck's inspired vision, within the purely mathematical 'confines' of *algebraic* geometry, was to replace 'space' (:topology) by sheaf cohomology

(MacLane and Moerdijk, 1992). Similarly, in Mallios' ADG, now within the field of *differential* geometry and with a strong inclination towards theoretical physics' applications (especially in QG) (Mallios, 1998a, 2005b), the ultimate desire is to do away with the background geometrical (smooth) spacetime (manifold) and the various (differential geometric) anomalies (:singularities and related unphysical infinities) that it carries (Mallios and Raptis, 2004; Raptis, 2005), and focus solely on the purely algebraico-categorical (:sheaf-theoretically modelled) relations between the 'objects' that 'live' on that surrogate background—i.e., the dynamical fields $\mathcal{D}$ and the laws (:differential equations) that they obey on their respective carrier sheaves $\mathcal{E}$, such as (6) (Mallios and Raptis, 2003; Mallios, 2003; Mallios and Raptis, 2004; Mallios, 2005a, 2004).

4.1. $\mathfrak{D}\mathfrak{T}_{fcq}$ as a Finitary Instanceof a GT

As noted in the introduction and briefly described in 3.1 above, Sorkin's main idea in Sorkin (1991) was to 'blow up' or 'smear' the points of the topological spacetime manifold X by 'fat' regions (:open sets) U about them belonging to locally finite open covers $\mathcal{U}_j$ of X, and then to replace (:approximate) the (locally) Euclidean $\mathcal{C}^0$-topology of X, which is supposed to be "*carried by its points*" (Sorkin, 1991), by T_0-fintoposets P_j.

Such an enterprize has a rather natural correspondent and quite a generalized description in category-theoretic terms. The latter pertains to Grothendieck's celebrated work on generalized topological spaces called *sites*, for the definition of which *covering sieves* (associated with open covers in the usual topological case), and a *Grothendieck topology* generated by them play a central role (MacLane and Moerdijk, 1992).[23] Thus, below we give a Grothendieck-type of description of Sorkin's 'finitarities,' which will subsequently help us view $\mathfrak{D}\mathfrak{T}_{fcq}$ as a GT of a finitary sort. In turn, in complete analogy to how the P_js in Sorkin's work were thought of as locally finite approximations of the continuous topology of X, here the EGT-like $\mathfrak{D}\mathfrak{T}_{fcq}$ can be regarded as a finitary substitute of the archetypical EGT $\mathcal{S}\mathbf{hv}^0(X)$—the topos of sheaves (of rings) of continuous functions on the topological manifold X. Thus, our research program of applying ADG-theoretic ideas to finitary spacetime and gravity (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004) is hereby reaching its categorical (:topos-theoretic) climax.

So to begin with, let X be the relatively compact region[24] of a topological manifold M that Sorkin considered in Sorkin (1991), and $\mathcal{U}_j$ (a locally finite) open cover for it, which also belongs to the inverse system (:topological refinement

[23] See Appendix B at the end for the relevant (abstract) definitions.

[24] Recall that a topological space X is said to be relatively compact if every open cover of it admits a locally finite refinement.

net) $\overleftarrow{\mathfrak{U}}_i := \{\mathcal{U}_j\}_{j\in I}$.[25] X may be viewed as a *poset category* $\mathcal{PO}(X)$, having for objects its open subsets and for (monic) arrows between them subset-inclusions (:one arrow for every pair of subsets, if they happen to be ordered by set-theoretic inclusion):[26]

$$U, V \subseteq X, \text{ open}: \ U \longrightarrow V \Leftrightarrow U \subseteq V \tag{13}$$

Then, a *sieve* S on U, $S(U)$, is an I-indexed collection of open subsets of U ($\{V_{i\in I} : V_i \longrightarrow U\}$) such that if $W \longrightarrow V \in S(U) \Rightarrow W \in S(U)$. One moreover says that $S(U)$ *covers* U (i.e., $S(U)$ is a *covering sieve* for the object U in $\mathcal{PO}(X)$), if $U \subseteq \cup_{i\in I} V_i \in S(U)$. Arrow-wise, one says that $S(U)$ *covers the arrow* $W \longrightarrow U$ in $\mathcal{PO}(X)$ when $W \longrightarrow \cup_i V_i$. With the aid of the relevant abstract definitions in appendix B, it is fairly straightforward to show for the concrete category $\mathcal{PO}(X)$ that:

Theorem: The collection $\{(U, S(U))\}$, as U runs through all the objects in $\mathcal{PO}(X)$, defines a Grothendieck topology J on $\mathcal{PO}(X)$.[27] Thence, the pair $(\mathcal{PO}(X), J)$ is an example of a *site*—the poset category $\mathcal{PO}X$ equipped with the Grothendieck topology J.

Equivalently, calling to action the open covering $\mathcal{U}_j$ of X (and *in extenso* of U, since plainly, $\cup_i V_i \longleftarrow U$), we can generate the following *covering sieve* for U based on $\mathcal{U}_j$:

$$S_j \equiv S_{\mathcal{U}_j}(U) = \{W \longrightarrow U : \ W \longrightarrow V_i, \text{ for some } V_i \in \mathcal{U}_j\} \tag{14}$$

It follows then that, as $\mathcal{U}_j$ runs through the inverse system (:refinement net) $\overleftarrow{\mathfrak{U}}_i = \{\mathcal{U}_j\}_{j\in I}$, a *basis* $\mathcal{B}_J$ for the said Grothendieck topology J on $\mathcal{PO}(X)$ is defined,[28] which also turns the said poset category into the *site* $(\mathcal{PO}, \mathcal{B}_J)$ (MacLane and Moerdijk, 1992).[29]

Now we have a good grasp of how Grothendieck-type of ideas can be applied to $\mathcal{PO}(X)$ so as to promote it to a site. Thus, let us turn to our category $\mathfrak{DT}_{fcq}$ and see how it can qualify as a finitary version of a GT-like structure.

Prima facie, and in view of the general and abstract ideas presented in appendix B, we could maintain that a 'natural' two-step path one could follow in order to cast $\mathfrak{DT}_{fcq}$ as a finitary type of GT is the following:

[25] In what follows, the reader should not confuse the refinement index 'j' used to label the fincovers in the topological refinement net, with the subscript 'i' labelling the open sets in a particular covering. However, we shall use the same symbol (:'I') to denote the index sets for both, hopefully without causing any misunderstanding.

[26] In fact, $\mathcal{PO}(X)$ is more than a poset, it is a *lattice*, but this will not concern us in what follows.

[27] This is just exercise 1 on page 155 of MacLane and Moerdijk (1992).

[28] Again, see Appendix B for the relevant (abstract) definitions.

[29] As noted in Appendix B, we hereby do not distinguish between the site $(\mathcal{PO}(X), J)$ and $(\mathcal{PO}(X), \mathcal{B}_J)$ generating it.

- first head-on endow $\mathfrak{DT}_{fcq}$ with some kind of *Grothendieck topology* thus turn it into a *site*-like structure, as we did for $\mathcal{PO}(X)$ above (MacLane and Moerdijk, 1992);
- then define *sheaves* over the resulting site and collect them into a GT-like structure.

However, the alert reader could immediately counter-observe that:

- On a first sight, it appears to be hopeless to directly try and Grothendieck-topologize the collection $\mathfrak{U}$ of *all* coverings $\mathcal{U}$ of the continuum X (or equivalently, the collection of all subtopologies $\tau_{\mathcal{U}}$ of X generated by them), since that family is not even a set proper—i.e., it is a *class*. As a result, if one wished to view $\mathfrak{U}$ as some sort of category, it would certainly *not* be small, unlike what the usual Grothendieck categories are assumed to be.[30]
- Moreover, as noted earlier, $\mathfrak{DT}_{fcq}$ is *not* a category of sheaves over a *fixed* base topological space, so that even if the latter was somehow Grothendieck-topologized to a *site*, $\mathfrak{DT}_{fcq}$ would still *not* be a GT proper. Rather, the base spaces of the fintriads in $\mathfrak{DT}_{fcq}$ are 'variable' entities, varying with the topological refinement (:coarse graining) of the underlying finitary coverings and their associated fintoposets.

Our way-out of this two-pronged impasse is based on the following two observations:

1. First, in response to the first 'dead-end' above, we note that the locally finite open covers $\mathcal{U}_i$ of Sorkin are, categorically speaking, 'good,' 'well behaved' objects when it comes to defining some generalized kind of 'topology' on them and taking 'limits' with respect to it. This is due to the fact that the collection $\overleftarrow{\mathfrak{U}}_i$ of all the finitary coverings of X comprise a so-called *cofinal* subset of the class $\mathfrak{U}$ of all (proper) open covers of X (Mallios, 1998a,b).
2. Second, and issuing from the point above, one could indeed use the topological refinement partial order ($\mathcal{U}_i \preceq \mathcal{U}_j \Leftrightarrow P_j \xrightarrow{f_{ji}} P_i$) on the net $\overleftarrow{\mathfrak{U}}_i$ (and its associated projective system $\overleftarrow{\mathfrak{P}}$ of fintoposets)[31] so as to define some kind of '*topological coarse graining sieve-topology*' on it *à la* Grothendieck. Then indeed, Sorkin's inverse limit 'convergence' of the

[30] See Appendix B. Similar reservations were expressed in Isham (1989), where the poset of subtopologies of a continuum appeared to be a class unmanageably large, hence unsuitable for quantization. Thus Isham had to resort to *finite topologies* (:topologies on a finite set of points) and the lattice of subtopologies thereof for a plausible quantization scenario.

[31] One can use $\overleftarrow{\mathfrak{U}}_i$ and $\overleftarrow{\mathfrak{P}}$ interchangeably, since one can transit from the $\mathcal{U}_i$s in $\overleftarrow{\mathfrak{U}}_i$ to the P_is in $\overleftarrow{\mathfrak{P}}$ by Sorkin's 'factorization algorithm' (7).

elements of $\overleftarrow{\mathfrak{U}}_i$ (and their associated P_is) to X at infinite topological refinement (10), can be accounted for on the grounds of that (abstract) topology. In the process however, a new type of GT arises, which we call '*a finitary approximation topos*' ('fat')—one that may be thought of as 'approximating' the usual 'continuum topos' $\mathcal{S}\mathbf{hv}^0(X)$ of sheaves of (rings of) continuous functions over the pointed $\mathcal{C}^0$-manifold X, much in the same way that the fintoposets P_i were seen to approximate the continuum X (or equivalently, the finsheaves in Raptis (2000b) were seen to approximate the 'continuum' sheaf $\mathcal{C}^0_X$).[32]

$\mathfrak{D}\mathfrak{T}_{fcq}$ as a 'fat'-Type of GT. We *can* endow the poset category $\overleftarrow{\mathfrak{U}}_i$ with a Grothendieck-type of topology by introducing the notion of *coarse graining finsieves*. Indirectly, these played a significant role earlier, when we defined the basis $\mathcal{B}_J$ for the site $(\mathcal{PO}(X), J)$.

So, recall that the fincovers in $\overleftarrow{\mathfrak{U}}_i$ are partially ordered by refinement, $\mathcal{U}_i \preceq \mathcal{U}_j$, which is tantamount to coarse graining continuous surjective maps (:arrows) between their respective fintoposets, $f_{ji} : P_j \longrightarrow f_i$ (9). With respect to these arrows, and with appendix B as a guide, we first define *coarse graining finsieves* $S_i \equiv S_{\mathcal{U}_i}$ covering each and every object (:fincover $\mathcal{U}_i$) in $\overleftarrow{\mathfrak{U}}_i$,[33] and from these we also define *coarse graining finsieves* S_{ji} covering each and every arrow $f_{ji} \in \overleftarrow{\mathfrak{U}}_i$. With S_i and S_{ji} in hand $(\forall \mathcal{U}_i,\ f_{ji} \in \overleftarrow{\mathfrak{U}}_i,\ i \in I)$, we then define a Gothendieck topology J_I on $\overleftarrow{\mathfrak{U}}_i$, thus converting it to a site: $(\overleftarrow{\mathfrak{U}}_i, J_I)$.[34] Parenthetically, as briefly alluded to earlier, the central projective limit result of Sorkin about $\overleftarrow{\mathfrak{P}}$ (10), may now be literally understood as the *'convergence' of the cofinal system $\overleftarrow{\mathfrak{U}}_i$ of finitary coverings, at the limit of their infinite topological refinement, to X relative to the Grothendieck-type of topology J_I (or the Grothendieck basis $(\mathcal{B}_i)_{i\in I}$) imposed on it.*

To unveil the GT-like character of $\mathfrak{D}\mathfrak{T}_{fcq}$, now that the net $\overleftarrow{\mathfrak{U}}_i$ of base spaces of its objects (:fintriads) has been Grothendieck-topologized, is fairly straightforward. We simply recall that $\mathfrak{D}\mathfrak{T}_{fcq}$ is a category of finsheaves of incidence algebras over Sorkin's fintoposets (:fintriads) deriving from the $\mathcal{U}_i$s in the Grothendieck net $\overleftarrow{\mathfrak{U}}_i$; hence, it is a finitary example of a GT (:a category of sheaves over a site). Moreover, because the inverse limit space of the P_is is effectively (i.e., modulo

[32] Note in this respect that $\mathcal{S}\mathbf{hv}^0(X)$ may indeed be thought of as a GT if we recall from above that $(\mathcal{PO}(X), J)$—or equivalently, $(\mathcal{PO}(X), \mathcal{B}_J)$—is a site. At the same time, $\mathcal{S}\mathbf{hv}^0(X)$ is a typical example of an ET as well (MacLane and Moerdijk, 1992).

[33] $S_i := \{\mathcal{U}_j \in \overleftarrow{\mathfrak{U}}_i : \mathcal{U}_j \preceq \mathcal{U}_i\}$.

[34] In fact, the covering coarse graining finsieves defined above determine (object and arrow-wise) a *basis* $\mathcal{B}_i$ for J_I (see Appendix B. As noted before, we think of $(\overleftarrow{\mathfrak{U}}_i, J_I)$ and $(\overleftarrow{\mathfrak{U}}_i, \mathcal{B}_i)_{i\in I}$ as being equivalent sites.

Hausdorff reflection) homeomorphic to X, and with our original regarding finsheaves as finitary approximations of $\mathcal{C}^0_X$ (for X a topological manifold), we may think of $\mathfrak{D}\mathfrak{T}_{fcq}$ as a 'fat' of (the EGT) $\mathcal{S}\mathbf{hv}^0(X)$—the category of sheaves of (rings of) continuous functions over the $\mathcal{C}^0$-manifold X.

$\mathfrak{D}\mathfrak{T}_{fcq}$ is Coherent and Localic. At this point it is important to throw in this presentation some technical remarks in order to emphasize that $\mathfrak{D}\mathfrak{T}_{fcq}$ is manifestly (i.e., by construction) *finitely generated*; hence, in a finitary sense, *coherent* (MacLane and Moerdijk, 1992). Moreover, by the way finsheaves were defined in Raptis (2000b) (i.e., as 'skyscraper'-like, fat/coarse *étale* spaces over the coarse, blown-up 'points' of X corresponding to the minimal open sets/nerves covering them relative to a locally finite cover $\mathcal{U}_i$ of X),[35] $\mathfrak{D}\mathfrak{T}_{fcq}$ *has enough points* and it is *localic* (MacLane and Moerdijk, 1992). Indeed, its underlying locale $\mathcal{L}oc$[36] is just the lattice of open subsets of X, while the points of X are recovered (modulo Hausdorff reflection) at the projective limit of infinite refinement of the base $\mathcal{U}_i$s (or their associated P_is) of the fintriads as we go along the coarse graining Grothendieck-type of sieve topology on $\mathfrak{D}\mathfrak{T}_{fcq}$.[37]

We can thus exploit the said 'localicality' of $\mathfrak{D}\mathfrak{T}_{fcq}$ in order to find out what is its subobject classifier Ω_{fcq}. We do this next.

4.2. The Subobject Classifier in the EGT $\mathfrak{D}\mathfrak{T}_{fcq}$

That $\mathfrak{D}\mathfrak{T}_{fcq}$ is localic points to a way towards its subobject classifier. One may think of the base spaces of the fintriads in $\mathfrak{D}\mathfrak{T}_{fcq}$ as '*finitary locales*' (finlocales) since, as noted earlier, they are the 'pointless' subtopologies τ_i of X generated by arbitrary unions of finite intersections of the open sets in each locally finite open cover $\mathcal{U}_i$ of X. We also noted that $\mathfrak{D}\mathfrak{T}_{fcq}$ can be regarded as a 'fat' of $\mathcal{S}\mathbf{hv}^0(X)$.[38] The generic object in the latter is $\mathcal{C}^0_X$—the sheaf of continuous functions on the pointed topological manifold X. When X is Grothendieck-topologized and turned into a site as described earlier, $\mathcal{C}^0_{(\mathcal{PO},J)}$ is a sheaf on a site and hence $\mathcal{S}\mathbf{hv}^0(\mathcal{PO}, J)$ *the* canonical example of an EGT (MacLane and Moerdijk, 1992).

[35] Indeed, as noted before, in Sorkin's work the points of X were substituted by $\overset{\mathcal{U}_i}{\sim}$-equivalence classes (and X by the corresponding P_is).

[36] A *locale* is a complete distributive lattice, otherwise known as a *Heyting algebra*. Locales are usually thought of as '*pointless topological spaces*.' It is a general result that every GT has an underlying locale (MacLane and Moerdijk, 1992).

[37] In the general case of an abstract coherent topos, there is a celebrated result due to Deligne stipulating roughly that *every coherent topos has enough points and its underlying locale is a topological space proper* (MacLane and Moerdijk, 1992).

[38] In this respect there is an intended metaphorical pun between the acronym 'fat' (:finitary approximation topos) and the epithet 'fat'. Indeed, the Grothendieck fintopos $\mathfrak{D}\mathfrak{T}_{fcq}$ associated with Sorkin's finitarities comes from substituting X's points by *fat*, coarse open regions about them (and hence the pointed X by the pointless finlocales τ_i).

Now, a central result in topos theory is the following: *for any sheaf on a site X, the lattice $\mathcal{L}oc$ of all its subsheaves is a complete Heyting algebra, a locale.*[39] Thus, in our case we just take $\mathcal{C}^0_X$ (which is a generic object in $\mathcal{S}\mathbf{hv}^0(X)$) for the said 'sheaf-on-a-site,' and the finsheaves (in the fintriads) for its subsheaves (:subobjects). Plainly then, the subobject classifier Ω in $\mathfrak{D}\mathfrak{T}_{fcq}$ is

$$\Omega(\mathfrak{D}\mathfrak{T}_{fcq}) = \mathcal{L}oc(\mathcal{C}^0_X) \equiv \mathcal{L}oc\big(\mathcal{C}^0_{(\mathcal{P}\mathcal{O}(X),J)}\big) \tag{15}$$

hence $\mathfrak{D}\mathfrak{T}_{fcq}$ is a localic topos, as anticipated above. In terms of covering sieves in the standard case of a topological space X like ours (again, regarded as a poset category $\mathcal{P}\mathcal{O}(X)$, with objects its open subsets U), we borrow verbatim from MacLane and Moerdijk (1992)[40] that "*for sheaves on a topological space with the usual open cover topology, the subobject classifier is the sheaf Ω on X defined by:* $\Omega(U) = \{V|\ V$ is open and $V \subset U\}$," or in terms of covering (:principal) sieves $S_\downarrow$ of lower sets for every V as above (i.e., $S_\downarrow(V) := \{V^{'}|\ V^{'} \subseteq V\}$), "$\Omega(U) = \{S_\downarrow(V)\}$" (*cf.* Appendix B).

Interpretational Matters: The Semantic Interplay Between Geometry and Logic in a Topos. One of the quintessential properties of a topos like $\mathcal{S}\mathbf{hv}^0(X)$ (for X a $\mathcal{C}^0$-manifold)—one that distinguishes it from the topos **Set** of '*constant sets*' (MacLane and Moerdijk, 1992)—is that its subobject classifier is a complete Heyting algebra, in contradistinction to the Boolean topos **Set** whose subobject classifier Ω is the Boolean binary alternative $\mathbf{2} = \{0, 1\}$. This inclines one to 'geometrically' interpret the former topos, in contradistinction to **Set**, as '*a generalized space of continuously variable sets, varying continuously with respect to the background continuum X*' (Lawvere, 1975; MacLane and Moerdijk, 1992). In the same semantic vain, we may interpret the variation of the objects living in $\mathfrak{D}\mathfrak{T}_{fcq}$ (:qausets (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2005; Mallios and Raptis, 2004)) as *entities varying with (topological) coarse graining*.

At the same time, every topos like $\mathcal{S}\mathbf{hv}^0(X)$ has not only a *geometrical*, but also a *logical* interpretation due to the non-Boolean character of its subobject classifier. Indeed, as noted before, Ω *can also be regarded as a generalized truth-values object*, the generalization being the transition from the Boolean truth values $\Omega = \mathbf{2} = \{0, 1\} = \{\top, \bot\}$ in **Set**, to a Heyting algebra-type of subobject classifier like the one in $\mathfrak{D}\mathfrak{T}_{fcq}$. This means that the so-called '*internal language*' (or logic) that can be associated with such topoi is (typed and) *intuitionistic*, in contrast to the 'classical,' Boolean logic of the topos **Set** of sets (MacLane and Moerdijk, 1992; Lambek and Scott, 1986).

[39] See theorem on page 146 in MacLane and Moerdijk (1992).
[40] Page 140.

In the last section we shall entertain the idea of exploring this close connection between geometry and logic in our particular case of interest (:$\mathfrak{D}\mathfrak{T}_{fcq}$), and we shall briefly pursue its physical implications and potential import to QG research.

However, for the time being, in the last paragraph of the present section we would like to give an ADG-based topos-theoretic presentation of topological refinement, which played a key role above in viewing $\mathfrak{D}\mathfrak{T}_{fcq}$ as a GT-like structure.

Finitary Differential Geometric Morphisms: Topological Refinement as a Natural Transformation from the Differential Geometric Standpoint of ADG. Regarding the *differential* geometric considerations that come hand in hand with ADG, since the fintriads encode not only topological, but also differential geometric structure, we may give a differential geometric flavor to Sorkin's purely topological acts of refinement in $\overleftarrow{\mathfrak{U}}_i$ and/or $\overleftarrow{\mathcal{P}}$.

We may recall from Section 2 that, from a general topos-theoretic vantage, a continuous map $f: X \longrightarrow Y$ between two topological spaces gives rise to a pair $\mathcal{GM}_f = (f_*, f^*)$ of covariant adjoint functors between the respective sheaf categories (:topoi) $\mathcal{S}\mathbf{hv}_X$ and $\mathcal{S}\mathbf{hv}_Y$ on them, called push-out (*alias*, direct image) and pull-back (*alias*, inverse image). In topos-theoretic jargon, $\mathcal{GM}$ is known as a *geometric morphism*. In our case of interest (:$\mathfrak{D}\mathfrak{T}_{fcq}$), the continuous surjection $f_{ji}: P_j \longrightarrow P_i$ (equivalently regarded as the map $f_{ji}: \tau_j \longrightarrow \tau_i$) corresponding to topological coarse graining (or equivalently, to covering refinement $\mathcal{U}_i \preceq \mathcal{U}_j$), induces via the Sorkin-Papatriantafillou scenario a pair $\mathcal{GM}_{f_{ji}} \equiv \mathcal{GM}_{ji} = (f_{ji*}, f_{ji}^*)$ of *fintriad morphisms* between the fintriads $\mathfrak{T}_i$ and $\mathfrak{T}_j$. $\mathcal{GM}_{ji}$ by definition (of differential triad morphisms) preserves the differential structure encoded in the finsheaves (of incidence algebras) comprising the corresponding fintriads, thus it may be called *finitary differential geometric morphism*. Thus, $\mathfrak{D}\mathfrak{T}_{fcq}$ may be perceived as a category whose objects are $\mathfrak{T}_i$s and whose arrows are $\mathcal{GM}_{ji}$s. The latter give a differential geometric slant to Sorkin's purely topological acts of refinement.

Furthermore, since the sheaves defining the fintriads are themselves functors[41] the functors (f_{ji*}, f_{ji}^*) between the general sheaf categories (of sets) $\mathcal{S}\mathbf{hv}_{\tau_i} \equiv \mathcal{S}\mathbf{hv}_i \ni \mathfrak{T}_i$ and $\mathcal{S}\mathbf{hv}_{\tau_j} \equiv \mathcal{S}\mathbf{hv}_j \ni \mathfrak{T}_j$ may be thought of as *natural* transformations, and hence $\mathfrak{D}\mathfrak{T}_{fcq}$ as a type of *functor category* (MacLane and Moerdijk, 1992). *In summa*, Sorkin's topological refinement may be understood in terms of ADG as a kind of *natural transformation of a differential geometric character*—we may thus coin it '*differential geometric refinement*.'

[41] Every sheaf (of any structures, e.g., sets, groups, vector spaces, rings, modules *etc.*) on a topological space may be identified with the (associated) *sheafification functor* between the respective categories (*e.g.*, from the category of topological spaces to that of groups) that produces it (Mallios, 1998a; MacLane and Moerdijk, 1992).

4.3. Functoriality: General Covariance is Preserved Under Refinement

Differential geometric refinement has a direct application and physical interpretation in ADG-gravity. In Mallios and Raptis (2003); Raptis (2005); Mallios and Raptis (2004) we saw how the Principle of General Covariance (PGC) of GR can be expressed categorically in ADG-theoretic terms as the $\mathbf{A}$-*functoriality* of the vacuum Einstein equations (6) or its finitary analogue (12). This means that (6) is expressed via the curvature $\mathcal{R}$, which is an $\mathbf{A}$-morphism or $\mathbf{A}$-tensor (where $\otimes_{\mathbf{A}}$ is the homological tensor product functor with respect to $\mathbf{A}$). The physical significance of the $\mathbf{A}$-functoriality of the ADG-theoretic vacuum gravitational dynamics is that our choice of field-measurements or field-coordinatizations encoded in $\mathbf{A}$ (*in toto*, our choice of $\mathbf{A}$), does not affect the field dynamics.[42] More familiarly, the ADG-analogue of the Diff(M)-implemented PGC of the differential manifold M based GR, is $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$—the principal (group) sheaf of field automorphisms. Since $\mathcal{E}$ is by definition locally coordinatized ('Cartesianly analyzed') into $\mathbf{A}$,[43] $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}|_{U\subset X} := \mathcal{E}nd\mathcal{E}(U)^{\bullet} = M_n(\mathbf{A})^{\bullet}(U) \equiv \mathcal{GL}(n,\mathbf{A})(U) \equiv \mathrm{GL}(n,\mathbf{A}(U))$, and the ADG-version of the PGC is (locally) implemented via $\mathcal{GL}(n,\mathbf{A})(U) \equiv \mathrm{GL}(n,\mathbf{A}(U))$—the (local) '$\mathbf{A}$-analogue' of the usual $GL(4,\mathbb{R})$ of GR standing for the group of general (local) coordinates' transformations.

Now, in Raptis (2005); Mallios and Raptis (2004) it was observed that the said generalized coordinates' $\mathbf{A}$-independence (:$\mathbf{A}$-functoriality) of the ADG-gravitational field dynamics has a rather natural categorical representation in terms of *natural transformations* (pun intended). Diagrammatically, this can be represented as follows:

$$\begin{array}{ccc}
\mathfrak{T}_1 \ni \mathbf{A}_1(:\mathcal{E}_1 \overset{\text{loc.}}{\simeq} \mathbf{A}_1^n) & \xrightarrow{\otimes_{\mathbf{A}_1}} & \mathcal{R}(\mathcal{E}_1) = 0 \\
\Big\downarrow \mathcal{N}_{\mathbf{A}} & & \Big\downarrow \mathcal{N}_{\mathcal{D}} \\
\mathfrak{T}_2 \ni \mathbf{A}_2(:\mathcal{E}_2 \overset{\text{loc.}}{\simeq} \mathbf{A}_2^n) & \xrightarrow[\otimes_{\mathbf{A}_2}]{} & \mathcal{R}(\mathcal{E}_2) = 0
\end{array}$$

and it reads that, changing (via the natural transformation $\mathcal{N}$) structure sheaves of algebras of generalized arithmetics (coordinates) from an $\mathbf{A}_1$ (and its

[42] In turn, in Mallios and Raptis (2003); Raptis (2005) and especially in Mallios and Raptis (2004), this $\mathbf{A}$-functoriality of the field dynamics was taken to support the ADG Principle of Field Realism (PFR): the connection field $\mathcal{D}$, expressed and partaking into the gravitational dynamics (6) via its curvature, is not 'perturbed' by our measurements/coordinatizations in $\mathbf{A}$.

[43] Recall from Section 2 that $\mathcal{E}$ is defined as a locally free $\mathbf{A}$-module of finite rank: $\mathcal{E} :\overset{\text{loc.}}{\simeq} \mathbf{A}^n$.

corresponding $\mathcal{E}_1$) to another $\mathbf{A}_2$ (and hence $\mathcal{E}_2$), the functorially, $\otimes_{\mathbf{A}}$-expressed (via the connection's curvature $\mathbf{A}$-morphism) vacuum Einstein equations remain 'form invariant.'[44]

The upshot here is that, the differential geometric refinement in $\mathfrak{D}\mathfrak{T}_{fcq}$ described above may be perceived as such a natural transformation-type of map (:finitary differential geometric morphism), as follows:

$$\begin{tikzcd}
\mathfrak{T}_j \ni \mathbf{A}_j(: \mathcal{E}_j \overset{\text{loc.}}{\simeq} \mathbf{A}_j^n) \arrow[r, "\otimes_{\mathbf{A}_j}"] \arrow[d, "\mathcal{N}_{ji\mathbf{A}} \equiv \mathcal{GM}_{ji\mathbf{A}}"'] & \mathcal{R}_j(\mathcal{E}_j) = 0 \arrow[d, "\mathcal{N}_{ji\mathcal{D}} \equiv \mathcal{GM}_{ji\mathcal{D}}"] \\
\mathfrak{T}_i \ni \mathbf{A}_i(: \mathcal{E}_i \overset{\text{loc.}}{\simeq} \mathbf{A}_i^n) \arrow[r, "\otimes_{\mathbf{A}_i}"'] & \mathcal{R}_i(\mathcal{E}_i) = 0
\end{tikzcd}$$

Thus, the ADG-version of the PGC of GR (:$\mathbf{A}$-functoriality) is preserved under such a not only topological, but also differential geometric, refinement. It is precisely this result that underlies the inverse system $\overleftarrow{\mathcal{E}}$ of vacuum Einstein equations and its continuum projective limit in the tower of inverse/direct systems of various finitary ADG-structures in expression (150) of Mallios and Raptis (2003) and/or (25) of Raptis (2005). In the latter paper especially, it is the projective limit of $\overleftarrow{\mathcal{E}}$ that was used to argue that the vacuum Einstein equations hold over the inner Schwarzschild singularity of the gravitational field of a point-particle both at the finitary ('discrete') and at the classical continuum inverse limit of infinite refinement (of the underlying base fintoposets of the fintriads involved).

5. EPILOGUE CUM SPECULATION: FOUR FUTURE QG PROSPECTS

In this rather lengthy concluding section we elaborate on the following four promising future prospects. First, on how one might further build on the EG-fintopos so as to incorporate 'quantum logical' ideas into our scheme. Then, we ponder on potential affinities between our ADG-based finitary EGT $\mathfrak{D}\mathfrak{T}_{fcq}$ and, (i) Isham's recent 'quantizing on a category' scenario, (ii) Christensen-Crane's recent causite theory, and (iii) Kock-Lawvere's SDG.

[44] Parenthetically, note in the diagram above that $\mathcal{N}$ has two indices: one for the 'object' (:structure sheaf $\mathbf{A}$), and one for the 'morphism' (:connection $\mathcal{D}$, or better, its $\mathbf{A}$-morphism curvature $\mathcal{R}$) in the respective differential triads $\mathfrak{T}_1$ and $\mathfrak{T}_2$ supporting them. By definition, a natural transformation between such sheaf (:functor) categories, is a functor that preserves objects (:sheaves) and their morphisms (:connections and their curvatures).

5.1. Representation Theory: Associated Hilbert Fintopos $\mathfrak{H}_{fcq}$

As noted earlier, by now it has been appreciated (primarily by mathematicians!) that a topos can be regarded both as a generalized space in a geometrical (e.g., topological) sense, as well as a generalized logical universe of variable set-like entities. Thus, in a topos, 'geometry' and 'logic' are thought of as being unified [45].

In our case, in view of this geometry-logic unification in a topos, a future prospect for further developing the theory is to relate the (differential) geometric (:'gravitational') information encoded in the fintopos $\mathfrak{D}\mathfrak{T}_{fcq}$, to the (internal) logic of an '*associated Hilbert fintopos*' $\mathfrak{H}_{fcq}$. The latter may be obtained from $\mathfrak{D}\mathfrak{T}_{fcq}$ in three steps:

1. From Aigner (1997); Stanley (1986); Zapatrin (1998) first invoke finite dimensional (irreducible) Hilbert space H_i matrix representations for every incidence algebra Ω_i dwelling in the stalks of every finsheaf $\mathbf{\Omega}_i$.
2. Then, like the corresponding incidence algebras were stacked into the finsheaves $\mathbf{\Omega}_i$, group the H_is into *associated* (Vassiliou, 2000, 2005) (:representation) Hilbert finsheaves $\mathcal{H}_i$ (again over Sorkin's fintoposets, which are subject to topological refinement).
3. Finally, organize the $\mathcal{H}_i$s into the fintopos $\mathfrak{H}_{fcq}$ as we did for the $\mathbf{\Omega}_i$s in $\mathfrak{D}\mathfrak{T}_{fcq}$, which may be fittingly coined the *Hilbert fintopos associated to* $\mathfrak{D}\mathfrak{T}_{fcq}$.

What one will have effectively obtained in the guise of $\mathfrak{H}_{fcq}$ is a *coarse graining presheaf of Hilbert spaces* (:a presheaf of Hilbert D-modules (Douglas, 1989; Kato, 1991, 2003; Kato and Struppa, 1999)) over the topological refinement poset category $\overleftarrow{\mathcal{P}}$ (or $\overleftarrow{\mathfrak{U}}_i$). To see this clearly, one must recall from Raptis and Zapatrin (2000, 2001); Zapatrin (2001a) that the correspondence $P_i \longrightarrow \Omega_i$ is a *contravariant functor* from the poset category $\overleftarrow{\mathcal{P}}$ and the continuous (:monotone) maps (:fintoposet morphisms) f_{ji} between the P_is, to a direct (:inductive) system $\overrightarrow{\Omega}$ of finitary incidence algebras and surjective algebra homomorphisms ω_{ji} between them. Such a contravariant functor may indeed be thought of as a *presheaf* (MacLane and Moerdijk, 1992).[45]

A 'Unified' Perspective on Geometrical and Logical Obstructions. The pair $(\mathfrak{D}\mathfrak{T}_{fcq}, \mathfrak{H}_{fcq})$ of fintopoi may provide us with strong clues on how to unify the 'warped' (gravitational) geometry and the 'twisted' (quantum) logic in a topos-theoretic setting. In this respect, the following analogy between the two topoi in the pair above is quite suggestive:

[45] This remark will prove to be useful in the next subsection.

- As we saw in Mallios and Raptis (2001, 2002, 2003), the finsheaves $\mathbf{\Omega}_i$ in $\mathfrak{D}\mathfrak{T}_{fcq}$ admit non-trivial (gravitational) connections $\mathcal{D}_i$, whose curvature $R_i(\mathcal{D}_i)$ measures some kind of *obstruction* preventing the following sequence of generalized differentials (:connections)

$$\begin{gathered}\mathbf{\Omega^0}(\mathcal{E}) \stackrel{\mathcal{D}\equiv\mathcal{D}^0}{\longrightarrow} \mathbf{\Omega^1}(\mathcal{E}) \stackrel{\mathcal{D}^1}{\longrightarrow} \mathbf{\Omega^2}(\mathcal{E}) \stackrel{\mathcal{D}^2}{\longrightarrow} \mathbf{\Omega^3}(\mathcal{E}) \stackrel{\mathcal{D}^3}{\longrightarrow} \cdots \\ \cdots \stackrel{\mathcal{D}^{i-1}}{\longrightarrow} \mathbf{\Omega^i}(\mathcal{E}) \stackrel{\mathcal{D}^i}{\longrightarrow} \mathbf{\Omega^{i+1}}(\mathcal{E}) \stackrel{\mathcal{D}^{i+1}}{\longrightarrow} \cdots\end{gathered} \tag{16}$$

from being *exact*. This is in contrast to the usual de Rham complex

$$\begin{gathered}\mathbf{0}(\equiv \mathbf{\Omega^{-2}}) \stackrel{\iota\equiv d^{-2}}{\longrightarrow} \mathbf{K}(\equiv \mathbf{\Omega^{-1}}) \stackrel{\epsilon\equiv d^{-1}}{\longrightarrow} \mathbf{A}(\equiv \mathbf{\Omega^0}) \\ \stackrel{d^0\equiv\partial}{\longrightarrow} \mathbf{\Omega^1} \stackrel{d^1\equiv d}{\longrightarrow} \mathbf{\Omega^2} \stackrel{d^2}{\longrightarrow} \cdots \mathbf{\Omega^n} \stackrel{d^n}{\longrightarrow} \cdots\end{gathered} \tag{17}$$

which is exact in our theory (:finitary de Rham theorem) (Mallios and Raptis, 2002). In other words, the curvature R of the connection $\mathcal{D}$ measures the departure of the latter from flatness, as opposed to ∂ which is flat.[46] Equivalently, in topos-theoretic parlance, the finsheaves in $\mathfrak{D}\mathfrak{T}_{fcq}$ do not have *global elements* (:sections) (MacLane and Moerdijk, 1992).[47] In $\mathfrak{D}\mathfrak{T}_{fcq}$, absence of global sections of its curved finsheaves is captured by the non-existence of arrows from the terminal object **1** in the topos to the said finsheaves. In ADG-theoretic terms (Mallios, 1998a,b; Mallios and Raptis, 2002), section-wise the obstruction (:departure from exactness) of the $\mathcal{D}$-complex above due to $R(\mathcal{D})$, may be expressed via the non-triviality of the '*global section functor*' and of the complex

$$\begin{gathered}\Gamma_X(\mathcal{S}^{\cdot}):\ \ \Gamma_X(\mathbf{0}) \longrightarrow \Gamma_X(\mathcal{S}^{\mathbf{0}}) \stackrel{\Gamma_X(d^0)}{\longrightarrow} \Gamma_X(\mathcal{S}^{\mathbf{1}}) \stackrel{\Gamma_X(d^1)}{\longrightarrow} \cdots \\ \cdots \stackrel{\Gamma_X(d^{n-1})}{\longrightarrow} \Gamma_X(\mathcal{S}^{\mathbf{n-1}}) \stackrel{\Gamma_X(d^n)}{\longrightarrow} \Gamma_X(\mathcal{S}^{\mathbf{n}}) \not\longrightarrow \cdots 0\end{gathered} \tag{18}$$

that it defines (Mallios, 1998a,b; Mallios and Raptis, 2002). Again, this is a fancy way of saying that the relevant vector (fin)sheaves ($\mathcal{S}_i^j \equiv \mathbf{\Omega}_i^j,\ j \in \mathbb{Z}_+$) do not admit global sections due to the non-triviality of $\mathcal{D}$. All this has been physically interpreted as *absence of global 'inertial' frames* (:'inertial observers') in ADG-fingravity (Mallios and Raptis, 2001).
- In a similar vain, but from a quantum logical standpoint, the associated Hilbert differential module finsheaves $\mathcal{H}_i$ do *not* admit global sections (:'valuation states')[48] in view of the Kochen-Specker theorem in standard quantum logic (Butterfield and Isham, 1998, 1999; Butterfield *et al.*,

[46] For example, section-wise in the relevant finsheaves: $(\mathcal{D}_i^{j+1} \circ \mathcal{D}_i^j)(s \otimes t) = t \wedge R_i(s)$, with $s \in \Gamma(U, \mathbf{\Omega}_i)$, $t \in \Gamma(U, \mathbf{\Omega^j})$ and U open in X. Thus, $R_i(\mathcal{D}_i)$ represents not only the measure of the departure from differentiating flatly, but also the deviation from setting up an (exact) cohomology sequence based on $\mathcal{D}_i$—altogether, a measure of the departure of $\mathcal{D}_i$ from (the) nilpotence (of ∂_i).

[47] Page 164.

[48] For dim $H_i > 2$.

2000; Butterfield and Isham, 2000). This is due to the well known fact that there are maximal Boolean subalgebras (:frames) of the the quantum lattice $\mathcal{L}_i(H_i)$ that are generated by mutually incompatible (:complementary, noncommuting) elements of $\mathcal{B}(H_i)$—the non-abelian C^*-algebra of bounded operators on $\mathcal{L}_i$ (whose hermitian elements are normally taken to represent quantum observables). The result is that certain presheaves (of sets) over the coarse graining poset of Boolean subalgebras of $\mathcal{L}_i(H_i)$ do not admit global sections. Logically, this is interpreted as saying that there are no global (Boolean) truth values in quantum logic, but only local ones (i.e., 'localized' at every maximal Boolean subalgebra or frame of $\mathcal{L}_i(H_i)$); moreover, the resulting 'truth values' space (:object) Ω in the corresponding presheaf topos ceases to be Boolean (:$\Omega = \mathbf{2}$) and becomes intuitionistic (:a Heyting algebra). In this sense, quantum logic is contextual (:'Boolean subalgebra localized') and 'neorealist' (:not Boolean like the classical logic of **Set**, but intuitionistic). Accordingly, in the aforesaid tetralogy of Isham *et al.*, it has been explicitly anticipated that *there must be a characteristic form that*, like $R_i(\mathcal{D}_i)$ above, *effectuates the said obstruction to assigning values to physical quantities globally over* $\mathcal{L}_i$.

- Thus, what behooves us in the future is to look for what one might call a '*quantum logical curvature*' characteristic form $\mathfrak{R}$ which measures *both* the (differential) geometrical obstruction in $\mathfrak{D}\mathfrak{T}_{fcq}$ to assigning global (inertial) frames at its finsheaves $\mathbf{\Omega}_i$, *and* the quantum logical obstruction to assigning global (Boolean) frames to their associated Hilbert finsheaves $\mathcal{H}_i$. This effectively means that one could attempt to bring together the intuitionistic (differential) geometric coarse graining in $\mathfrak{D}\mathfrak{T}_{fcq}$, with the also intuitionistic (quantum) logical coarse graining in $\mathfrak{H}_{fcq}$.

One might wish to approach this issue of logico-geometrical obstructions in a unified algebraic way. For instance, one could observe that both the differential geometric obstruction (in GR) and the quantum logical obstruction (in QM) above are due to some non-commutativity in the basic 'variables' involved, in the following sense:

- the differential geometric obstruction, represented by the curvature characteristic form, is due to the non-commutativity of covariant derivations (:connections); while,
- the quantum logical obstruction is ultimately due to the existence of noncommuting (complementary) quantum observables such as position (:x) and momentum (:∂_x).

Parenthetically, we note in this line of thought that for quite some time now the idea has been aired that the 'macroscopic' non-commutativity of covariant derivatives in the curved spacetime continuum of GR is due to a more fundamental 'microscopic' quantum non-commutativity in a 'discrete,' dynamical quantum logical (:'quantal') substratum underlying it.[49] For instance, in Selesnick (1991, 1994, 1995, 2004) one witnesses how the gravitational curvature form of a spin-Lorentzian (:$SL(2, \mathbb{C})$-valued) connection arises 'spontaneously' (as a coherent state condensate) from a Schwinger-type of dynamical variational principle of basic bivalent spinorial quantum-time atoms (:'chronons') teeming the said reticular and quantal substratum coined the 'quantum net' (Finkelstein, 1988, 1989, 1991, 1996). In this model we can quote Finkelstein from the prologue of Finkelstein (1996) maintaining that "*logics come from dynamics*." For similar ideas, but in a topos-theoretic setting, see Raptis (1996).

On the other hand, in ADG-gravity there is no such fundamental distinction between a (classical) continuum and a (quantal) discretum spacetime. All there exists and is of import in the theory are the algebraic (dynamical) relations between the ADG-fields $(\mathcal{E}, \mathcal{D})$ themselves, without dependence on an external (to those fields) surrogate background space(time), be it 'discrete/quantal' or 'continuous/classical' (Mallios and Raptis, 2001, 2002, 2003, 2004; Raptis, 2005). Thus in our ADG-framework, if we were to investigate deeper into the possibility that some sort of quantum commutation relations are ultimately responsible for the aforementioned obstructions, we should better do it 'sheaf cohomologically'—i.e., in a purely algebraic manner that pays respect to the fact that ADG is not concerned at all with the geometrical structure of a background spacetime, but with the algebraic relations of the 'geometrical objects' that live on that physically fiducial base. The latter are nothing else than the connection fields $\mathcal{D}$ and the sections of the relevant sheaves $\mathcal{E}$ that they act on, while at the same time sheaf cohomology is *the* technical (:algebraic) machinery that ADG employs from the very beginning of the *aufbau* of the theory (Mallios, 1998a,b, 2005b; Mallios and Raptis, 2002).

We thus follow our noses into the realm of the ADG-perspective on geometric (pre)quantization and second quantization (Mallios, 1998a,b, 1999, 2004; Mallios and Raptis, 2002; Mallios, 2005b) in order to track the said obstructions in $(\mathfrak{D}\mathfrak{T}_{fcq}, \mathfrak{H}_{fcq})$ down to algebraic, *sheaf cohomological commutation relations*. What we have in mind is to propose some sheaf cohomological commutation relations between certain characteristic forms that uniquely characterize the finsheaves (and the connections acting on them) in the fintopos $\mathfrak{D}\mathfrak{T}_{fcq}$; while, by the functoriality of geometric pre- and second quantization *à la* ADG (Mallios, 2001, 2005b, 1999, 2004; Mallios and Raptis, 2002), to transfer these characteristic forms and their algebraic commutation relations to their associated Hilbert finsheaves in $\mathfrak{H}_{fcq}$.

The following discussion on how we might go about and set up the envisaged sheaf cohomological commutation relations is tentative and largely heuristic.

[49] David Finkelstein in private e-mail correspondence (2000).

One can begin by recalling some basic ('axiomatic') assumptions in ADG-field theory (Mallios, 1998a,b; Mallios and Raptis, 2002, 2003, 2004):

- The fields (*viz.* connections) $\mathcal{D}$ exist 'out there' independently of us—the observers or 'measurers' of them (Principle of Field Realism in Mallios and Raptis (2003, 2004)). Recall from Section 2 that in ADG-field theory, by a *field* we refer to the pair $(\mathcal{E}, \mathcal{D})$. The connection $\mathcal{D}$ is the '*proper*' part of the field, while the vector sheaf $\mathcal{E}$ is its *representation* (*alias*, carrier or action) space.
- Various collections $\mathcal{U}_i$ of covering open subsets of the base topological space X are the *systems of local open gauges*.
- Our measurements (of the fields) take values in the structure sheaf $\mathbf{A}$ of generalized arithmetics (coordinates or coefficients) that *we* choose in the first place, with $\mathbf{A}(U)$ (for a U in some $\mathcal{U}_i$ chosen) the *local coordinate gauges* that we set up for (i.e., to measure) the fields.
- From a geometric pre- and second quantization vantage (Mallios, 1998b, 1999, 2004; Mallios and Raptis, 2002, 2003; Mallios, 2005b), our field-measurements correspond to *local (particle) coordinatizations of the fields*. They are the ADG-analogues of '*particle position measurements*' of the fields. Local position (particle) states are represented by local sections of the representation (:associated) sheaves $\mathcal{E}$,[50] which in turn are by definition locally ($\mathbf{A}_U$-) isomorphic to $\mathbf{A}^n$ (i.e., $\mathcal{E}(U) \equiv \mathcal{E}|_U \simeq (\mathbf{A}(U))^n \equiv (\mathbf{A}|_U)^n$). Accordingly, given a local gauge U in a chosen gauge system $\mathcal{U}_i$, the collection $e^U = \{(U; e_1, \ldots, e_n)\}$ of local sections of $\mathcal{E}$ on U is called a *local frame* (or local gauge basis) of $\mathcal{E}$. Any section $s \in \Gamma(U, \mathcal{E}) \equiv \mathcal{E}(U)$ can be written as a linear combination of the e_αs above, with coefficients in $\mathbf{A}(U)$.
- As noted before, $\mathcal{E}$ is the carrier or action space of the connection. $\mathcal{D}$ acts on the local particle (coordinate-position) states (i.e., the local sections) of $\mathcal{E}$ and changes them. Thus, the (flat) sheaf morphisms ∂, and *in extenso* the curved ones $\mathcal{D}$, are the generalized (abstract), ADG-theoretic analogues of *momenta*.

With these abstract semantic correspondences:

$$\begin{aligned} &1.\ \text{abstract position/particle states} \longrightarrow \text{local sections of } \mathcal{E} \\ &2.\ \text{abstract momentum/field states} \longrightarrow \text{local expression of } \mathcal{D} \end{aligned} \tag{19}$$

we are in a position to identify certain characteristic forms that could engage into the envisaged (local) quantum commutation relations (relative to a chosen family $\mathcal{U}$ of local gauges):

[50] Furthermore, with respect to the spin-statistics connection, local boson states are represented by local sections of *line* sheaves (:vector sheaves of rank 1), while fermions by local sections of vector sheaves of rank greater than 1 (Mallios, 1998a,b, 1999, 2004; Mallios and Raptis, 2002, 2003).

1. Concerning the abstract analogue of ‘particle/position’ (:the part $\mathcal{E}$ of the ADG-field pair $(\mathcal{E}, \mathcal{D})$), we might consider the so-called coordinate 1-cocycle $\phi_{\alpha\beta} \in \mathcal{A}ut\mathcal{E} = GL(n, \mathbf{A}(U_{\alpha\beta})) = \mathcal{GL}(n, \mathbf{A})(U_{\alpha\beta})$ (for $U_{\alpha\beta} = U_\alpha \cap U_\beta;\ U_\alpha, U_\beta \in \mathcal{U}$), which completely characterizes (and classifies) sheaf cohomologically the vector sheaves $\mathcal{E}$ (Mallios, 1998a,b; Mallios and Raptis, 2002; Mallios, 2005b).[51] What we have here is an instance of the age-old Kleinian dictum that local states (:‘geometry’) of $\mathcal{E}$—i.e., the (local) sections that comprise it,[52] are *how* they transform (here, under changes of local gauge $\phi_{\alpha\beta}$).[53]
2. Concerning the abstract analogue of ‘field/momentum’ (:the part $\mathcal{D}$ of the ADG-field pair $(\mathcal{E}, \mathcal{D})$), we might consider the so-called gauge potential $\mathcal{A}$ of the connection $\mathcal{D}$, which completely determines $\mathcal{D}$ locally.[54] With respect to $\mathcal{U}$, $\mathcal{A}_{ij}$ is (locally) a 0-cochain of (local) $n \times n$ matrices with entries from (:local sections in) $\mathbf{\Omega}^1(U)$ ($U \in \mathcal{U}$; n is the rank of $\mathcal{E}$). In other words, for a given system (frame) of local gauges $\mathcal{U} = (U_\alpha)_{\alpha \in I}$, $\mathcal{A}_{ij}^{(\alpha)} \in C^0(\mathcal{U}, M_n(\Omega) = C^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})$—i.e., $\mathcal{A}$ is (locally) an endomorphism (:$\mathcal{E}nd\mathcal{E}$) valued ‘1-form.’[55]

In line with the above, we may thus posit (locally) the following abstract (pre)quantum commutation relations between the generalized (:abstract) ‘*position characteristic form*’ $\phi_{\alpha\beta}$ and the generalized (:abstract) ‘*momentum characteristic form*’ $\mathcal{A}_{ij}$:[56]

$$[\phi|_U, \mathcal{A}|_U] \propto \mathfrak{R} \in \mathcal{E}nd\mathcal{E}(U) \tag{20}$$

which make sense (i.e., they are well defined), since ϕ (locally) takes values in $\mathcal{A}ut\mathcal{E}(U) = \mathcal{GL}(n, \mathbf{A})(U)$, while $\mathcal{A}$ in $\mathcal{E}nd\mathcal{E}$, which both allow for (the definition of a Lie-type of) a product like the commutator.

[51] Indeed, we read from Mallios (1998b) for example in connection with the Picard cohomological classification of the vector sheaves involved in ADG, that “*any vector sheaf $\mathcal{E}$ on X is uniquely determined (up to an* $\mathbf{A}$*-isomorphism) by a coordinate* 1*-cocycle, say,* $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathbf{A}))$, *associated with any local frame $\mathcal{U}$ of $\mathcal{E}$*.”

[52] And recall *the* epitome of sheaf theory, namely, that *a sheaf is its (local) sections*. That is, the entire sheaf space $\mathcal{E}$ can be (re)constructed (by means of restriction and collation) from its (local) sections. Local information (:sections) is glued together to yield the ‘total sheaf space.’

[53] Another way to express this Kleinian viewpoint, $\mathcal{E}$ is the associated (:representation) sheaf of the ‘symmetry’ group sheaf $\mathcal{A}ut\mathcal{E} = \mathcal{GL}(n, \mathbf{A})$ of its self-transmutations. Equivalently, the particle states (:local sections of $\mathcal{E}$) of the field carry a representation of the symmetry group of field automorphisms. Here, the epithet ‘symmetry’ pertains to the fact that $\mathcal{A}ut\mathcal{E}$ is the symmetry group sheaf of vacuum Einstein ADG-gravity (6), implementing our abstract version of the PGC of GR.

[54] Indeed, we read from Mallios (1998b) that “*$\mathcal{D}$ is determined (locally) uniquely by $\mathcal{A}$*.” Recall also that $\mathcal{D}$ locally splits as $\partial + \mathcal{A}$ (Mallios, 1998a,b; Mallios and Raptis, 2003).

[55] In this respect, one may recall that in the usual theory (CDG of smooth manifolds), the gauge potential is (locally) a Lie algebra-valued 1-form.

[56] With indices omitted.

What behooves us now is to give a physical interpretation to the commutator above according to the ADG-field semantics. Loosely, (20) is the relativistic and covariant ADG-gravitational analogue of the Heisenberg uncertainty relations between the position and momentum 'observables' of a (non-relativistic) quantum mechanical particle (or, *in extenso*, of a relativistic quantum field). One should highlight here a couple of things concerning (20):

- First a mathematical observation: the functoriality between $\mathfrak{DT}_{fcq}$ (gravity; differential geometry) and $\mathfrak{H}_{fcq}$ (quantum theory; quantum logic) carries the characteristic forms and their uncertainty relations from the former to the latter.
- Second, a physical observation: the '*self-quantumness*' of the ADG-field $(\mathcal{E}, \mathcal{D})$. As it has been stressed many times in previous work (Mallios and Raptis, 2003; Raptis, 2005; Mallios and Raptis, 2004), the ADG-field $(\mathcal{E}, \mathcal{D})$ is a dynamically autonomous, 'already quantum' and in need of no formal process of quantization. The autonomy pertains to the fact that there is no background geometrical spacetime (continuum or discretum) interpretation of the purely algebraic, dynamical notion of ADG-field (:ADG-gravity is a genuinely background independent theory). Moreover, the field is 'self-quantum' (or 'self-quantized') as its two constituent parts—$\mathcal{E}$ and $\mathcal{D}$—engage into the quantum commutation relations (20), while its background spacetime independence entails that in our scheme *quantization of gravity is not dependent on or does not entail quantization of spacetime itself*.
- Since in ADG-gravity there is no background spacetime (continuous or discrete) interpretation, while all is referred to the algebraic (dynamical) relations in sheaf space, *there is no spacetime scale dependence of the ADG-expressed law of vacuum Einstein gravity* (6), *or of the commutator* (20). Recalling from the introduction our brief remarks about the 'conspiracy' of the equivalence principle of GR and the uncertainty principle of QM, which apparently prohibits the infinite localization of the gravitational field past the so-called Planck space-time length-duration without creating a black hole; by contrast, *in ADG-gravity the Planck space-time is not thought of as a fundamental 'obstruction'—an unavoidable regularization cut-off scale—to infinite localization beyond which the classical continuum spacetime gives way to a quantal discretum one*. As noted in Mallios and Raptis (2003, 2004); Raptis (2005), the vacuum Einstein equations hold both at the classical continuum (6) and at the quantal discontinuum level (12), and they are not thought of as breaking down below Planck scale.
- If any 'noncommutativity' is involved in ADG-gravity (say, *à la* Connes (Connes, 1994)), it is encoded in $\mathcal{A}ut\mathcal{E}$ ('field foam' (Mallios and Rosinger, 2001)), or anyway, in $\mathcal{E}nd\mathcal{E}$ where ϕ and $\mathcal{A}$ take their values. That is, in our

scenario, if any kind of 'noncommutativity' is involved, it pertains to the dynamical self-transmutations of the field (:$\mathcal{D}$) and its 'inherent' quantum particle states (:local sections of $\mathcal{E}$) (Mallios and Raptis, 2004).[57]

- Since the sheaf cohomological quantum commutation relations (20) are preserved by topological (or differential geometric) refinement and are carried intact to the 'classical continuum limit' (Raptis and Zapatrin, 2000, 2001), we may interpret the usual (differential geometric) curvature obstruction (in $\mathfrak{D}\mathfrak{T}_{fcq}$) as some kind of 'macroscopic quantum effect' (coming from $\mathfrak{H}_{fcq}$), like Finkelstein has intuited for a long while now.[58]

5.2. Potential Links with Isham's Quantizing on a Category

A future project of great interest is to relate our fintopos-theoretic labors on ADG-fingravity above with Isham's recent '*Quantizing on a Category*' (QC) general mathematical scheme (Isham, 2003b, 2004a,b, 2005).

On quite general grounds, the algebraico-categorical QC is closely akin to ADG both conceptually and technically, having affine basic motivations and aims. For example, QC's main goal is to quantize systems with configuration (or history) spaces consisting of 'points' having internal (algebraic) structure. The main motivation behind QC is the grave failure of applying the conventional quantization concepts and techniques to 'systems' (e.g., causets or spacetime topologies) whose configuration (or general history) spaces are far from being structureless-pointed differential (:smooth) manifolds. Isham's approach hinges on two innovations: first it regards the relevant entities as objects in a category, and then it views the categorical morphisms as abstract analogues of momentum (derivation maps) in the usual (manifold based) theories. As it is also the case with ADG, although this approach includes the standard manifold based quantization techniques, it goes much further by making possible the quantization of systems whose 'state' spaces are not pointed-structureless smooth continua.

As hinted to above, there appear to be close ties between QC and ADG-gravity—ties which ought to be looked at closer in the future. *Prima facie*, both schemes concentrate on evading the (pathological) point-like base differential manifold—be it the configuration space of some classical or quantum physical system, or the background spacetime arena of classical or quantum (field) physics—and they both employ 'pointless,' categorico-algebraic methods. Both focus on an abstract (categorical) representation of the notion of derivative or derivation: in QC Isham abstracts from the usual continuum based notion of

[57] Recall that in ADG-gravity the PGC of GR is modelled after $\mathcal{A}ut\mathcal{E}$, while from a geometric prequantization viewpoint, the local quantum particle states of the ADG-gravitation field (:the local sections of $\mathcal{E}$) are precisely the ones that are 'shuffled around' by $\mathcal{A}ut\mathcal{E}$—the states on which $\mathcal{D}$ acts to dynamically change.

[58] See footnote 46 above.

vector field (derivation), to arrive at the categorical notion of arrow field which is a map that respects the internal structure of the categorical objects one wishes to focus on (e.g., topological spaces or causets); while in our work, the notion of derivative is abstracted and generalized to that of an algebraic connection, defined categorically as a sheaf morphism, on a sheaf of suitably algebraized structures (e.g., causets or finitary topological spaces and the incidence algebras thereof).

A key idea that could potentially link QC with our fintopos $\mathfrak{DT}_{fcq}$ for ADG-fingravity (and with ADG in general) is that in the former, as a result of a formal process of quantization developed there, a *presheaf of Hilbert spaces* (of variable dimensionality) arises as a 'induced representation space' of the so-called '*category quantization monoid of arrow-fields*' defined by the arrow-semigroup of the base category C that one chooses to work with.[59] The crux of the argument here is that this presheaf is similar to the coarse 'graining Hilbert presheaf' that the associated (:representation) Hilbert fintopos $\mathfrak{H}_{fcq}$ was seen to determine above. This similarity motivates us to wish to apply Isham's QC technology to our particular case of interest in which the base category is $\mathfrak{DT}_{fcq}$ and the arrows between them the coarse graining fintriad geometric morphisms, taking also into account the internal structure of the finsheaves involved. In this respect, perhaps also the sheaf cohomological quantization algebra envisaged above can be related to the category (monoid) quantization algebras engaged in QC. All in all, we will have in hand a particular application of Isham's QC scenario to the case of our ADG-perspective on Sorkin's fintoposets, their incidence algebras, and the finsheaves (:fintriads) thereof in $\mathfrak{DT}_{fcq}$.

5.3. Potential Links with the Christensen-Crane Causites

As noted in the introduction, recently there has been a Grothendieck categorical-type of approach to quantum spacetime geometry and to non-perturbative Lorentzian QG called *causal site* (:causite) theory (Christensen and Crane, 2004). Causite theory bears a close resemblance in both motivation and technical (:categorical) means employed with our $fcqv$-ADG-gravity—in particular, with the present topos-theoretic version of the latter. Here are some common features:

- Both employ general homological algebra (:category-theoretic) ideas and techniques. Causite theory may be perceived as a 'categorification' and quantization of causet theory, while our scheme may be understood as the 'sheafification' and (pre)quantization of causets.

[59] Roughly, as briefly mentioned above, the objects of C in Isham's theory represent generalized (:abstract) 'configuration states,' while the transformation-arrows (:morphisms) between them, analogues of momentum (:derivation) maps.

- In both approaches, simplicial ideas and techniques are central. In causite theory there are two main structures, both of which are modelled after partial orders: the topological and the causal. The relevant categorical structures of interest are bisimplicial 2-categories. As a result, the Grothendieck-type of topos structure envisaged to be associated with (pre)sheaves (of Hilbert spaces) over causites is a 2-topos (:bitopos). In our approach on the other hand, the causal and topological structures of the world are supposed to be physically indistinguishable, hence they 'collapse' into a single (simplicial) partial order. This subsumes our main position that *the physical topology is the causal topology* (Mallios and Raptis, 2001). As a result, the fintopos $\mathfrak{D}\mathfrak{T}_{fcq}$-organization of the finsheaves of qausets over Sorkin's finsimplicial complexes (:fintoposets) accomplished herein is a Grothendieck-type of 'unitopos,' not a bitopos.
- We read from Christensen and Crane (2004): "*A very important feature of the topology of causal sites is that they have a tangent* 2*-bundle, which is analogous to the tangent bundle of a manifold*." In the purely algebraic ADG-gravity, we are not interested in such a geometrical interpretation and conceptual imagery (:base spacetime, tangent space, tangent bundle etc). Presumably, one would like to have a tangent bundle-like structure in one's theory in order to identify its sections with 'derivation maps,' thus have in one's hands not only *topological*, but also *differential* structure. Having differential geometric structure on causites, then one would like "*to impose Einstein's equation*" (as a *differential* equation proper!) "*on a causal site purely intrinsically*". Moreover, in Christensen and Crane (2004) it is observed that, in general, "*doing sheaf theory over such generalized spaces (:sites) is an important part of modern mathematics*." In ADG-fingravity, we do most (if not all) of the above *entirely algebraically*, *a fortiori* without any geometrical commitment to a background '*space(time)*'—be it discrete or continuous.
- Last but not least, both approaches purport to be inherently finitistic and hence *ab initio* free from singularities and other unphysical infinities. The ultimate aim (or hope!) of causite theory is to "*lead to a description of quantum physics free from ultraviolet divergences, by eliminating the underlying point set continuum*" (Christensen and Crane, 2004). So is ADG-gravity's (Mallios and Raptis, 2001, 2002, 2003, 2004; Raptis, 2005).

It would certainly be worthwhile to investigate closer the conceptual and technical affinities between causite theory and $fcqv$-ADG-gravity.[60]

[60] Louis Crane in private e-correspondence.

5.4. Potential Links with Kock-Lawvere's SDG

From a purely mathematical perspective, but with applications to QG also in mind, it would be particularly interesting to see how can one carry out under the prism of our EG-fintopos $\mathfrak{D}\mathfrak{T}_{fcq}$ the basic finitary, ADG-theoretic constructions *internally* in the said fintopos by using the 'esoteric,' intuitionistic-type of language (:logic) of this topos exposed in this paper. For example, by stepping into the constructive world of the topos, one could bypass the 'problem' (because **A**-functoriality-violating) of defining *derivations* in ADG.[61] *En passant*, as briefly alluded to in footnote 4 and in the previous subsection, the reader will have already noticed that no notion of '*tangent vector field*' is involved in ADG—i.e., no maps in $Der : \mathbf{A} \longrightarrow \mathbf{A}$ are defined, as in the classical geometrical manifold based theory (CDG). Loosely, this can be justified by the fact that in the purely algebraic ADG the (classical) geometrical notion of 'tangent space' to the (arbitrary) simply topological base space X involved in the theory, has essentially no meaning, but more importantly, *no physical significance*, since X itself plays no role in the (gravitational) equations defined as differential equations proper via the derivation-free ADG-machinery.

Even more importantly for bringing together ADG and SDG, and having delimited the topos-theoretic (:intuitionistic-logical) background underlying both ADG and SDG, one can then compare the notion of *connection*—arguably, *the* key concept that actually qualifies either theory as being a *differential* geometry proper—as this concept appears in a categorical guise in both theories (Kock and Reyes, 1979; Kock, 1981; Lavendhomme, 1996; Mallios, 1998a,b, 2005b; Vassiliou, 1994, 1999, 2005). For the definition of the synthetic differential (:connection) ∂, the intuitionistic internal logic of the 'formal smooth topoi' involved plays a central role, dating back to Grothendieck's stressing the importance of 'rings with nilpotent elements' in the context of *algebraic* geometry. At the same time, for the definition of ∂ as a sheaf morphism in ADG, no serious use has so far been made of the intuitionistic internal logic of the sheaf categories in which the relevant sheaves live. One should thus wait for an explicit construction of those sheaves from within (i.e., by using the internal language of) the relevant topoi. This is a formidable task well worth exploring; for, applications' wise, recall again for instance from the previous subsection the following words from Christensen and Crane (2004):

> ". . . As yet, we do not know how to impose Einstein's equation on a causal site purely intrinsically . . . "

On the other hand, we certainly know how to (and we actually do!) impose (6)—or its finitary version (12)—from within the objects (:finsheaves) comprising $\mathfrak{D}\mathfrak{T}_{fcq}$, but without having made actual use of the latter's internal logic.

[61] Chris Mulvey in private e-correspondence.

To wrap up the present paper, we would like to recall from the conclusion of Stachel (1993)[62] the following 'prophetic' exchange between Abraham Fränkel and Albert Einstein:[63]

> "... In December 1951 I had the privilege of talking to Professor Einstein and describing the recent controversies between the (neo-) intuitionists and their 'formalistic' and 'logicistic' antagonists; *I pointed out that the first attitude would mean a kind of atomistic theory of functions, comparable to the atomistic structure of matter and energy. Einstein showed a lively interest in the subject and pointed out that to the physicist such a theory would seem by far preferable to the classical theory of continuity. I objected by stressing the main difficulty, namely, the fact that the procedures of mathematical analysis, e.g., of differential equations, are based on the assumption of mathematical continuity, while a modification sufficient to cover an intuitionistic-discrete medium cannot easily be imagined. Einstein did not share this pessimism and urged mathematicians to try to develop suitable new methods not based on continuity*[64] ..."

A modern-day version of the words above, which also highlights the close affinity between sheaf and topos theory (MacLane and Moerdijk, 1992) *vis-à-vis* QT and QG, is due to Selesnick:[65]

> "... One of the primary technical hurdles which must be overcome by any theory that purports to account, on the basis of microscopic quantum principles, for macroscopic effects (such as the large-scale structure of what appears to us as space-time, i.e., gravity) is the handling of the transition from 'localness' to 'globalness.' In the 'classical' world this kind of maneuver has been traditionally effected either measure-theoretically—by evaluating largely mythical integrals, for instance—or geometrically, through the use of sheaf theory, which, surprisingly, has a close relation to topos theory. The failure of integration methods in traditional approaches to quantum gravity may be ascribed in large measure to the inappropriateness of maintaining a manifold—a 'classical' object—as a model for space-time, while performing quantum operations everywhere else. If we give up this classical manifold and replace it by a quantal structure, then the already considerable problem of mediating between local and global (or micro and macro) is compounded with problems arising from the appearance of subtle effects like quantum entanglement, and more generally by the problems arising from the non-objective nature of quantum 'reality' ..."

[62] With the original citation being (Fränkel, 1954).

[63] This author wishes to thank John Stachel for timely communicating (Stachel, 1993) to him. This quotation, with an extensive discussion on how it anticipates our application of ADG-ideas to finitistic QG, as well as the former's potential links with SDG, can be found in Mallios and Raptis (2004).

[64] Our emphasis.

[65] Steve Selesnick (private correspondence).

Most of the discussion in this long epilogue has been highly speculative, largely heuristic, tentative and incomplete, thus it certainly requires further elaboration and scrutiny. However, we feel that further advancing our theory on those four QG research fronts in the near future is well worth the effort.

APPENDIX A: THE DEFINITION OF AN ABSTRACT ELEMENTARY TOPOS

This appendix is used in 3.2 to show that $\mathfrak{D}\mathfrak{T}_{fcq}$ is a finitary example of an ET in the sense of Lawvere and Tierney (MacLane and Moerdijk, 1992). To recall briefly this formal and abstract mathematical structure,[66] a small category[67] $\mathfrak{C}$ is said to be an ET if it has the following properties:

- $\mathfrak{C}$ is closed under finite limits. Equivalently, $\mathfrak{C}$ is said to be *finitely complete*. As noted earlier, categorical limits are also known as projective (inverse) limits, thus a topos $\mathfrak{C}$ is defined to be closed under projective limits.
- $\mathfrak{C}$ is *cartesian*. That is, for any two objects A and B in $\mathfrak{C}$, one can form the object $A \times B$—their cartesian product. All such finite products are supposed to be 'computable' in $\mathfrak{C}$ (:$\mathfrak{C}$ is closed under finite cartesian products).
- $\mathfrak{C}$ has an *exponential structure*. This essentially means that for any two objects $A, B \in \mathfrak{C}$, one can form the object B^A consisting of all arrows (in $\mathfrak{C}$) from A to B. As noted earlier, the set B^A is usually designated by $\mathcal{H}om(A, B)$ (:'hom-sets of arrows'), and it is supposed to effectuate the following canonical isomorphisms for an arbitrary object C in $\mathfrak{C}$ relative to the cartesian product structure: $\mathcal{H}om(C \times A, B) \simeq \mathcal{H}om(C, B^A)$ (or equivalently: $B^{C\times A} \simeq (B^A)^C$).
- $\mathfrak{C}$ has a *subobject classifier* object Ω. This means that for any object A in $\mathfrak{C}$, its subobjects (write sub(A)) canonically correspond to arrows from it to Ω: $\text{sub}(A) \simeq \mathcal{H}om(A, \Omega) \equiv \Omega^A$.

A couple of secondary, 'corollary' properties of a topos $\mathfrak{C}$ are:

- $\mathfrak{C}$ is also *finitely cocomplete*. That is, $\mathfrak{C}$ is also closed under finite inductive (direct) limits. Thus *in toto*, a topos $\mathfrak{C}$ is defined to be *finitely bicomplete* (co-complete or co-closed).
- $\mathfrak{C}$ has a preferred object $\mathbf{1}$, called the *terminal object*, over which all the other objects in $\mathfrak{C}$ are 'fibered'. That is, for any $A \in \mathfrak{C}$, there is a unique morphism $A \longrightarrow \mathbf{1}$.

[66] For more technical details, the reader is referred to MacLane and Moerdijk (1992).

[67] A category is said to be small if the families of objects and arrows that constitute it are proper sets—i.e., not classes (MacLane and Moerdijk, 1992).

- Dually, $\mathfrak{C}$ also possesses a so-called *initial object* $\mathbf{0}$ which is 'included' in each and every object of $\mathfrak{C}$; write: $\mathbf{0} \longrightarrow A, \ (\forall A \in \mathfrak{C})$.
- Finally, again dually to the fact that a topos $\mathfrak{C}$ has (finite) products, it also has (finite) coproducts.[68]

Due to its possessing (i) finite cartesian products, (ii) exponentials, and (iii) a terminal object, an ET $\mathfrak{C}$ is said to be a *cartesian closed category*, an equivalent denomination (MacLane and Moerdijk, 1992). Let it be noted here that the primary definitional axioms for an ET above are not minimal. Indeed, a small category $\mathfrak{C}$ need only possess finite limits, a subobject classifier Ω, as well as a so-called power object $PB = \Omega^A$ (for every object $A \in \mathfrak{C}$), in order to qualify as an ET proper. Then, the rest of the properties outlined above can be derived from these three basic ones (MacLane and Moerdijk, 1992).

APPENDIX B: THE DEFINITION OF AN ABSTRACT GROTHENDIECK TOPOS

This appendix is used in 4.1 to show that $\mathfrak{D}\mathfrak{T}_{fcq}$ is a GT (MacLane and Moerdijk, 1992). 4To recall briefly this formal and abstract mathematical structure,[69] a small category $\mathfrak{C}$ is said to be a GT if the following two conditions are met:

- There is a base category $\mathfrak{B}$ endowed with a so-called *Grothendieck topology* on its arrows. $\mathfrak{B}$, thus topologized, is said to be a *site*; and
- Relative to $\mathfrak{B}$, $\mathfrak{C}$ is a *sheaf category*—i.e., it is a category of sheaves over the site $\mathfrak{B}$.

Let us elaborate a bit further on these two defining features of an abstract GT.

Grothendieck Topologies: Sites. There are two (equivalent) definitions of a Grothendieck topology on a category $\mathfrak{B}$, which we borrow from MacLane and Moerdijk (1992). Both use the notion of a *sieve*—in particular, of so-called *covering sieves*. *Prima facie*, the use of covering sieves in defining a Grothendieck topology is tailor-cut for $\mathfrak{D}\mathfrak{T}_{fcq}$, which follows Sorkin's tracks in Sorkin (1991), since we saw in 3.1 that the notions of *open coverings* and *sieve-topologies* generated by them play a central role in Sorkin's fintoposet scheme.

[68] For example, in the category **Set** of sets—the archetypical example of a topos that other topoi aim at generalizing—the coproduct is the disjoint union (or direct sum) of sets and it is usually denoted by $\coprod$ (or $\bigoplus$). On the other hand, in the category of (commutative) rings, or of $\mathbb{K}$-algebras, or even of sheaves of such algebraic objects, the coproduct is the usual tensor product $\otimes_{\mathbb{K}}$ (while the product remains the cartesian product, as in the universe **Set** of structureless sets).

[69] Again, for more technical details, the reader can refer to MacLane and Moerdijk (1992).

Thus, for an object A in a category $\mathfrak{B}$, a *sieve* S on A (write $S(A)$) is a set of arrows (:morphisms) $f : * \longrightarrow A$ in $\mathfrak{B}$[70] such that for all arrows $g \in \mathfrak{B}$ with $\mathrm{dom}(f) = \mathrm{ran}(g)$,[71]

$$f \in S(A) \Longrightarrow f \circ g \equiv fg \in S(A)$$

That is, S is a *right ideal* in $\mathfrak{B}$, when the latter is viewed as an associative arrow-semigroup with respect to morphism-multiplication (:arrow concatenation).

Parenthetically, in the case of a topological space X,[72] regarded as a *poset category* $\mathcal{PO}(X)$ of its open subsets $U \subseteq X$ and having as (*monic*) morphisms between them open subset-inclusions (i.e., $\forall$ open $U, V \subseteq X : V \longrightarrow U \Leftrightarrow V \subseteq U$), a sieve on U is a *poset ideal* (with '$\subseteq$' the relevant partial order).

From the definition of a sieve above, it follows that if $S(A)$ is a sieve on A in $\mathfrak{B}$, and $g : B \longrightarrow A$ any arrow with $\mathrm{ran}(g) = A$, then the collection

$$g^*(S) = \{\mathfrak{B} \ni h : \mathrm{ran}(h) = B,\ gh \in S\}$$

is also a sieve on B called the pull-back (sieve) of (the sieve) S along (the arrow) g.

Having defined sieves, an abstract kind of topology J—the so-called *Grothendieck topology*—can be defined on a general category $\mathfrak{B}$ in terms of them. Thus, J is an assignment to every object A in $\mathfrak{B}$ of a family $J(A)$ of sieves on A, satisfying the following three properties:[73]

- **Maximality:** the *maximal* sieve $m(A) = \{f : \mathrm{ran}(f) = A\}$ belongs to $J(A)$;
- **Stability:** if $S \in J(A)$, then $g^*(S)$ belongs to $J(B)$ for any arrow g as above;
- **Transitivity:** if $S \in J(A)$ and $T(A)$ is any sieve on A such that $\forall g$ as above, $g^*(T) \in J(B)$, then $T \in J(A)$.

We say that S *covers* A (or that S is a *covering sieve* for A relative to J on $\mathfrak{B}$), when it belongs to $J(A)$. Also, we say that a sieve S *covers the arrow* $g : B \longrightarrow A$ above, if $g^*(S) \in J(B)$.

With these two 'covering' definitions, and by identifying the objects of $\mathfrak{B}$ by their identity arrows $i_A : A \longrightarrow A$ ($\forall A \in \mathfrak{B}$), the three defining properties of a Grothendieck topology on $\mathfrak{B}$ above can be recast in '*arrow-form*' as follows (MacLane and Moerdijk, 1992):

70 $*$ stands for an arbitrary object in $\mathfrak{B}$, which happens to be the domain of an arrow $f \in S(A)$.

71 Where 'dom' and 'ran' denote the '*domain*' and '*range*' maps on the arrows of $\mathfrak{B}$, respectively. That is, for $\mathfrak{B} \ni h : B \longrightarrow C$, $\mathrm{dom}(h) = B$ and $\mathrm{ran}(h) = C$.

72 In which we are interested in 4.1 in connection with Sorkin's 'finitarities' in Sorkin (1991) and ours in $\mathfrak{D}\mathfrak{T}_{fcq}$.

73 The following is the '*object-form*' definition of a Grothendieck topology (MacLane and Moerdijk, 1992). Its equivalent '*arrow-form*' follows shortly.

- **Maximality:** if S is a sieve on A and $f \in S$, then S covers f;
- **Stability:** if S covers an arrow $f : B \longrightarrow A$, it also covers $g \circ f$, $\forall g : C \longrightarrow B$; and,
- **Transitivity:** if S covers the arrow f above, and T is a sieve on A covering all the arrows in S, then T covers f.

Finally, an instrumental notion (used in 4.1) is that of a *basis* $\mathcal{B}_J$ or *generating set of morphisms* for a (covering sieve in a) Grothendieck topology J on a general category $\mathfrak{B}$ with pullbacks. Following (MacLane and Moerdijk, 1992), $\mathcal{B}_J$ is an assignment to every object $A \in \mathfrak{B}$ of a collection $\mathcal{B}_J(A) := \{f : \mathrm{ran}(f) = A\}$ of arrows in $\mathfrak{B}$ with range A, enjoying the following properties:

- **Iso-Maximality:** every isomorphism in $\mathfrak{B}$, with range A, belongs to $\mathcal{B}_J(A)$;
- **Stability:** for a family $F = \{f_i : B_i \longrightarrow A \ (i \in I)\}$ in $\mathcal{B}_J(A)$, and any morphism $g : C \longrightarrow A$, the family of pullbacks $\{f_i^* : B_i \times_A C \longrightarrow C\}$ along each f_i belongs to $\mathcal{B}_J(C)$; and,
- **Transitivity:** for F as above, and for each $i \in I$ one has another family of arrows $G_j = \{g_{ij} : C_{ij} \longrightarrow B_i \ (j \in I_i)\}$ in $\mathcal{B}_J(B_i)$, the family $F \circ G := \{f_i \circ g_{ij} : C_{ij} \longrightarrow A \ (i \in I, j \in I_i)\}$ also belongs to $\mathcal{B}_J(A)$.

A category $\mathfrak{B}$ equipped with a Grothendieck topology J as defined above is called a *site*. A site is usually symbolized by the pair $(\mathfrak{B}, J)$. If instead of J one has prescribed a basis $\mathcal{B}_J$ on $\mathfrak{B}$, by slightly abusing terminology, the pair $(\mathfrak{B}, \mathcal{B}_J)$ can still be called a site—namely, it is the site generated by the *covering families* of arrows in $\mathcal{B}_J(A)$ $(\forall A \in \mathfrak{B})$.

In summa, a site represents a generalized topological space on which (abstract) sheaves can be defined. Indeed, as noted in the main text, Grothendieck invented sites in order to develop generalized *sheaf cohomology* theories thus be able to tackle various problems in algebraic geometry (MacLane and Moerdijk, 1992).

Sheaves on a Site: GT. With a site $(\mathfrak{B}, J)$ in hand, an abstract GT is defined to be a category $\mathfrak{C}$ of sheaves over a base site. One writes symbolically, $\mathfrak{C} := \mathcal{S}\mathbf{hv}(\mathfrak{B}, J)$.
• It is a general fact that every GT is an ET (MacLane and Moerdijk, 1992).[74]

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[74] Page 143.

be of importance to QG research from the plethora of promising mathematical physics ideas that ADG is pregnant to—in this paper in particular, about the potentially close ties between ADG and topos theory *vis-à-vis* QG. Numerous exchanges with Georgios Tsovilis on the potential topos-theoretic development of ADG-gravity are also acknowledged. Maria Papatriantafillou is also gratefully acknowledged for e-mailing to this author her papers on the categorical aspects of ADG (Papatriantafillou, 2000, 2001, 2003a,b), from which some beautiful *LaTeX* commutative diagrams were obtained. Finally, he wishes to acknowledge financial support from the European Commission in the form of a European Reintegration Grant (ERG-CT-505432) held at the University of Athens, Greece.

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